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Centre de Recherches Mathématiques
Université de Montréal

Free Random Variables

D. V. Voiculescu
K. J. Dykema
A. Nica

American Mathematical Society

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Free Random Variables

A noncommutative probability approach to free products with applications to random matrices, operator algebras and harmonic analysis on free groups

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CHAPTER 1

Free Products

1.1. Groups

The free product of groups is the universal object defined in the following way.

DEFINITION 1.1.1. Let $(G_\iota)_{\iota\in I}$ be a family of groups. The *group free product* of this family, denoted $*_{\iota\in I} G_\iota$, is the unique (up to isomorphism) group G together with homomorphisms $\psi_\iota\colon G_\iota \to G$ such that given any group H and homomorphisms $\phi_\iota\colon G_\iota \to H$ there exists a unique homomorphism $\Phi = *_{\iota\in I}\phi_\iota\colon G \to H$ making the diagram in Figure 1.1 commute. (The homomorphisms, ψ_ι, are injective.).

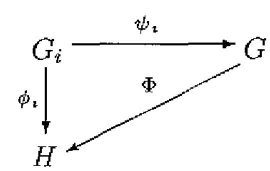

FIGURE 1.1

We may construct the free product by taking the set of reduced words,

$$G = \{\, g_1 g_2 \cdots g_n \mid g_j \in G_{\iota_j} \setminus \{e\} \text{ with } \iota_1 \neq \iota_2 \neq \iota_3 \neq \cdots \neq \iota_n \,\} \cup \{\varnothing\}$$

with multiplication defined as juxtaposition followed by reducing. Thus

$$(g_1 \cdots g_n)(g_1' \cdots g_m') = \text{ reduced word of } g_1 \cdots g_n g_1' \cdots g_m'.$$

1.2. Unital algebras

The free product for unital algebras is a universal object defined in a manner similar to that for groups.

DEFINITION 1.2.1. If $(A_\iota)_{\iota\in I}$ is a family of unital algebras, then their *unital algebra free product* $*_{\iota\in I} A_\iota$ is the unique unital algebra A together with unital homomorphisms $\psi_\iota\colon A_\iota \to A$ such that given any unital algebra B and unital homomorphisms $\phi_\iota\colon A_\iota \to B$ there exists a unique unital homomorphism $\Phi = *_{\iota\in I}\phi_\iota\colon A \to B$ making the diagram in Figure 1.2 commute.

As a vector space, the free product $*_{\iota\in I} A_\iota$ is the quotient of the vector space which has as basis the set

$$B = \{\, a_1 a_2 \cdots a_n \mid n \in \mathbb{N},\ a_j \in A_{\iota_j},\ \iota_1 \neq \iota_2 \neq \cdots \neq \iota_n \,\}$$