



# CRM MONOGRAPH SERIES

Centre de Recherches Mathématiques  
Université de Montréal

## Free Random Variables

D. V. Voiculescu  
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A. Nica



American Mathematical Society

Volume 1



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## Free Random Variables

A noncommutative probability approach  
to free products with applications  
to random matrices, operator algebras  
and harmonic analysis on free groups

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**American Mathematical Society**  
Providence, Rhode Island, USA

The production of this volume was supported in part by the Chaire André Aisenstadt, the Fonds pour la Formation de Chercheurs et l'Aide à la Recherche (Fonds FCAR), and the Natural Sciences and Engineering Research Council of Canada (NSERC). The second author was supported in part by the John and Fannie Hertz Foundation.

2000 *Mathematics Subject Classification*. Primary 46-XX;  
Secondary 46L10, 46L35, 47B80.

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**Library of Congress Cataloging-in-Publication Data**

Voiculescu, D. V. (Dan V.), 1949-

Free random variables: a noncommutative probability approach to free products with applications to random matrices, operator algebras, and harmonic analysis on free groups / D. V. Voiculescu, K. J. Dykema, A. Nica.

p. cm.

Includes bibliographical references.

ISBN 0-8218-6999-X (alk. paper)

1. Operator algebras. 2. Selfadjoint operators. 3. Free products (Group theory)

I. Dykema, K. J., 1967-. II. Nica, A., 1961-. III. Title.

QA326.V65 1992

512'.55—dc20

92-37964

CIP

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Printed in the United States of America.

Reprinted with corrections 2002

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This publication was typeset using  $\text{\LaTeX}$ ,  
the American Mathematical Society's  $\text{\TeX}$  macro system,  
and submitted to the American Mathematical Society in camera-ready  
form by the Centre de Recherches Mathématiques.

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10 9 8 7 6 5 4 3 2 07 06 05 04 03 02

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## Free Products

### 1.1. Groups

The free product of groups is the universal object defined in the following way.

DEFINITION 1.1.1. Let  $(G_i)_{i \in I}$  be a family of groups. The *group free product* of this family, denoted  $*_{i \in I} G_i$ , is the unique (up to isomorphism) group  $G$  together with homomorphisms  $\psi_i: G_i \rightarrow G$  such that given any group  $H$  and homomorphisms  $\phi_i: G_i \rightarrow H$  there exists a unique homomorphism  $\Phi = *_{i \in I} \phi_i: G \rightarrow H$  making the diagram in Figure 1.1 commute. (The homomorphisms,  $\psi_i$ , are injective.).

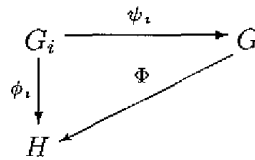


FIGURE 1.1

We may construct the free product by taking the set of reduced words,

$$G = \{g_1 g_2 \cdots g_n \mid g_j \in G_{i_j} \setminus \{e\} \text{ with } i_1 \neq i_2 \neq i_3 \neq \cdots \neq i_n\} \cup \{\emptyset\}$$

with multiplication defined as juxtaposition followed by reducing. Thus

$$(g_1 \cdots g_n)(g'_1 \cdots g'_m) = \text{reduced word of } g_1 \cdots g_n g'_1 \cdots g'_m.$$

### 1.2. Unital algebras

The free product for unital algebras is a universal object defined in a manner similar to that for groups.

DEFINITION 1.2.1. If  $(A_i)_{i \in I}$  is a family of unital algebras, then their *unital algebra free product*  $*_{i \in I} A_i$  is the unique unital algebra  $A$  together with unital homomorphisms  $\psi_i: A_i \rightarrow A$  such that given any unital algebra  $B$  and unital homomorphisms  $\phi_i: A_i \rightarrow B$  there exists a unique unital homomorphism  $\Phi = *_{i \in I} \phi_i: A \rightarrow B$  making the diagram in Figure 1.2 commute.

As a vector space, the free product  $*_{i \in I} A_i$  is the quotient of the vector space which has as basis the set

$$B = \{a_1 a_2 \cdots a_n \mid n \in \mathbb{N}, a_j \in A_{i_j}, i_1 \neq i_2 \neq \cdots \neq i_n\}$$