Applied Stochastic Processes

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(last update: July 18, 2007)

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1 Discrete Markov chains

Markov processes form an important class of random processes with many applications in areas like physics, biology, computer science or finance. The characteristic property of a Markov process is its lack of memory, that is, the decision where to go next may (and typically does) depend on the current state of the process but not on how it got there. If the process can take only countably many different values then it is referred to as a *Markov chain*.

1.1 Basic properties and examples

A stochastic process $X = (X_t)_{t\in T}$ is a random variable which takes values in some *path space* $\mathcal{S}^T := \{x = (x_t)_{t\in T} : T \to \mathcal{S}\}$. Here, the space of possible outcomes $\mathcal{S}$ is some discrete (i.e., finite or countably infinite) *state space* and X_t is the state at time t (with values in $\mathcal{S}$). In this chapter, we will assume that time is discrete, i.e., we take the index set T to be the non-negative integers $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$.

The distribution of X can be specified by describing the dynamics of the process, i.e., how to start and how to proceed. By the multiplication rule, we have

$$\begin{aligned}
&\mathbb{P}\{(X_0, X_1, \ldots, X_n) = (x_0, x_1, \ldots, x_n)\} \\
&\quad = \mathbb{P}\{X_0 = x_0\}\, \mathbb{P}\{X_1 = x_1 \,|\, X_0 = x_0\}\, \mathbb{P}\{X_2 = x_2 \,|X_0 = x_0, X_1 = x_1\} \cdots \\
&\qquad\qquad \cdots\ \mathbb{P}\{X_n = x_n \,|\, X_0 = x_0, \ldots, X_{n-1} = x_{n-1}\} \\
&\quad =: p_0(x_0)\, p_1(x_0, x_1) \cdots p_n(x_0, \ldots, x_n\}. \qquad (1.1)
\end{aligned}$$

The left-hand side of equation (1.1) can be written as:

$$\mathbb{P}\{X \in B_{x_0,\ldots,x_n}\}, \quad \text{where } B_{x_0,\ldots,x_n} = \{x_0\} \times \{x_1\} \times \cdots \times \{x_n\} \times \mathcal{S}^{\{n+1,n+2,\ldots\}}. \qquad (1.2)$$

Note that the functions p_j, $j \geq 0$ have to satisfy the following conditions.

1. $p_j(x_0, \ldots, x_j) \geq 0$ for all $j \geq 0$, $x_0, \ldots, x_j \in \mathcal{S}$;
2. $\sum_{x_j\in\mathcal{S}} p_j(x_0, \ldots, x_j) = 1$ for all $j \geq 0$, $x_0, \ldots, x_{j-1} \in \mathcal{S}$.

Remark. The measures in (1.1) uniquely extend to a probability measure on $(\mathcal{S}^{\mathbb{N}_0}, \mathcal{B})$, where $\mathcal{B}$ is the σ-algebra generated by all sets of the form (1.2) (see Theorem 3.1 in [1]).

In general, the p_j may depend on the entire collection $x_0, \ldots, x_j$. However, little can be said about interesting properties of a stochastic process in this generality. This is quite different for so-called Markovian stochastic dynamics, where the p_j depend on x_{j-1} and x_j only.

Definition 1.1 *Let $\mathcal{S}$ be a countable space and $P = (P_{xy})_{x,y\in\mathcal{S}}$ a stochastic matrix (i.e., $P_{xy} \geq 0$ and $\sum_{y\in\mathcal{S}} P_{xy} = 1$, $\forall\, x \in \mathcal{S}$). A sequence of $\mathcal{S}$-valued random variables (r.v.'s) $X_0, X_1, \ldots$ is called a* Markov chain *(MC) with* state space $\mathcal{S}$ *and* transition matrix P*, if*

$$\mathbb{P}\{X_{n+1} = y \,|\, X_0 = x_0, \ldots, X_{n-1} = x_{n-1}, X_n = x\} = P_{xy} \qquad (1.3)$$

holds for every $n \in \mathbb{N}_0$ *and* $x_0, \ldots, x_{n-1}, x, y \in \mathcal{S}$ *(provided that the conditional probability is well defined).*

Remarks.

- To be precise, the process $(X_n)_{n\geq 0}$ defined above is a *time-homogeneous* Markov chain (in general, the matrix P in (1.3) may depend on n).

- The dynamics (described by P) will be considered as fixed, but we will vary the initial distribution $\mu := \mathbb{P}\{X_0 \in \cdot\,\}$. It is standard notation to add the initial distribution as a subscript:

$$\begin{aligned}\mathbb{P}_\mu\{X_0 = x_0, \ldots, X_n = x_n\} &:= \mu(x_0)\,\mathbb{P}\{X_1 = x_1, \ldots, X_n = x_n | X_0 = x_0\} \\ &= \mu(x_0)\,P_{x_0 x_1} \cdots P_{x_{n-1} x_n}. \qquad (1.4)\end{aligned}$$

  If $\mu = \delta_z$, then we write $\mathbb{P}_z := \mathbb{P}_{\delta_z}$ for short.

The following proposition formalizes the Markov property, which says that the future and the past are conditionally independent given the present state of the chain.

Proposition 1.2 (Markov property) *Let* $(X_n)_{n\geq 0}$ *be a (time-homogeneous) Markov chain with state space* $\mathcal{S}$ *and initial distribution* μ*, then*

$$\mathbb{P}_\mu\{(X_m, \ldots, X_{m+n}) \in B \,|\, X_m = x,\, (X_0, \ldots, X_{m-1}) \in B'\} = \mathbb{P}_x\{(X_0, \ldots, X_n) \in B\}$$

holds for every $m, n \in \mathbb{N}_0$, $x \in \mathcal{S}$, $B \subset \mathcal{S}^{n+1}$ *and* $B' \subset \mathcal{S}^m$*.*

In other words, conditional on the event $\{X_m = x\}$ the process $(X_{m+n})_{n\geq 0}$ is a Markov chain started at x, independent of $(X_0, \ldots, X_{m-1})$.

Proof. By (1.4), we have for any $x_0, \ldots, x_{m-1}, x, y_0, \ldots, y_n$

$$\begin{aligned}&\mathbb{P}_\mu\{X_0 = x_0, \ldots, X_{m-1} = x_{m-1}, X_m = x, X_m = y_0, X_{m+1} = y_1, \ldots, X_{m+n} = y_n\} \\ &= \delta_{x y_0}\,\mu(x_0) P_{x_0 x_1} \cdots P_{x_{m-1} x}\, P_{y_0 y_1} \cdots P_{y_{n-1} y_n} \\ &= \mathbb{P}_\mu\{X_0 = x_0, \ldots, X_m = x\}\,\mathbb{P}_x\{X_0 = y_0, \ldots, X_n = y_n\}. \qquad (1.5)\end{aligned}$$

Summation over all $(x_0, \ldots, x_{m-1}) \in B'$ and $(y_0, \ldots, y_n) \in B$ gives

$$\begin{aligned}&\mathbb{P}_\mu\{(X_0, \ldots, X_{m-1}) \in B',\, X_m = x, (X_m, \ldots, X_{m+n}) \in B\} \\ &= \mathbb{P}_\mu\{(X_0, \ldots, X_{m-1}) \in B',\, X_m = x\}\,\mathbb{P}_x\{(X_0, \ldots, X_n) \in B\}. \qquad (1.6)\end{aligned}$$

Dividing either side by $\mathbb{P}_\mu\{(X_0, \ldots, X_{m-1}) \in B',\, X_m = x\}$ gives the assertion of the proposition.

A simple consequence of the Markov property is the following formula for the n-step transition probabilities of the Markov chain $(X_n)_{n\geq 0}$.

Lemma 1.3 (n-step transition probabilities) *Let $(X_n)_{n\geq 0}$ be a Markov chain with state space $\mathcal{S}$ and transition matrix P. Then, for every $x, y \in \mathcal{S}$ and every $n \in \mathbb{N}_0$*

$$\mathbb{P}_x\{X_n = y\} = P^n_{xy}$$

holds, where

$$P^n = (P^n_{xy})_{x,y\in\mathcal{S}} = \underbrace{P\cdots P}_{n\ times},\ n \geq 1$$

and $P^0 := Id$.

Proof. By induction. For $n = 0$ we have

$$\mathbb{P}_x\{X_0 = y\} = \delta_{xy} = Id_{xy} = P^0_{xy}.$$

To get from n to $n+1$ note that by means of the law of total probability we have

$$\begin{aligned}\mathbb{P}_x\{X_{n+1} = y\} &= \sum_{z\in\mathcal{S}} \mathbb{P}_x\{X_n = z\}\,\mathbb{P}_x\{X_{n+1} = y \,|\, X_n = z\} \\ &= \sum_{z\in\mathcal{S}} P^n_{xz}\,P_{zy} \;=\; P^{n+1}_{xy},\end{aligned}$$

where for the second equality we have used the induction hypothesis and the Markov property (Proposition 1.2).

Remark. If the chain is started with initial distribution μ then the law of total probability and Lemma 1.3 give

$$\begin{aligned}\mathbb{P}_\mu\{X_n = y\} &= \sum_{x\in\mathcal{S}} \mu(x)\,\mathbb{P}_x\{X_n = y\} \\ &= \sum_{x\in\mathcal{S}} \mu(x)\,P^n_{xy} \;=:\; (\mu P^n)(y),\ y \in \mathcal{S}.\end{aligned}$$

Examples.

- **Random products**. Let $Y_0, Y_1, \ldots$ be independent and identically distributed (i.i.d.) random variables with values in a discrete set $\mathcal{S} \subset \mathbb{R}$. Set

  $$X_n := Y_0 \cdots Y_n,\ n \geq 0.$$

  and

  $$X'_n := Y_{n-1}Y_n,\ n \geq 1 \text{ and } X'_0 := 1.$$