

# **Theory and Applications of Stochastic Differential Equations**

**ZEEV SCHUSS**

*Department of Mathematical Sciences  
Tel-Aviv University, Ramat-Aviv  
Israel*

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# Review of Probability Theory

## 1.1 EVENTS AND SAMPLE SPACES

Consider the experiment of tossing a fair coin three times. The possible outcomes of this experiment are HHH, HHT, HTH, THH, HTT, THT, TTH, and TTT, where H denotes heads and T denotes tails. Each possible outcome of the experiment is called an *elementary event*. Thus there are eight elementary events in the experiment of tossing a coin three times. More complicated events can be expressed as combinations of the elementary events. Thus the event “two or more heads turn up” in the coin-tossing experiment, which we denote by  $D$ , consists of the elementary events HHH, HHT, HTH, THH; that is,

$$D = \{HHH, HHT, HTH, THH\}.$$

The set  $\Omega$  of all elementary events corresponding to an experiment is called a *sample space* and each elementary event, denoted by  $\omega$ , is called a *point* in  $\Omega$ . In the particular example under consideration, we have

$$\Omega = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}.$$

We write  $\omega \in \Omega$  to read “ $\omega$  is a point in (or an element of)  $\Omega$ .” Any event  $A$  that consists of elementary events is a subset of  $\Omega$ . In particular, the impossible event  $\emptyset$ , that is, an event that contains no elementary events, is called *the empty set*. Let  $A$  and  $B$  be events in  $\Omega$ ; then we say that  $A$  is a *subset of  $B$*  if every element of  $A$  is also an element of  $B$ , and we write  $A \subset B$ . Obviously,  $A \subset \Omega$ ,  $\emptyset \subset A$ , and  $A \subset A$ . For example, the set  $D$  in the coin-tossing experiment is a subset of the set (event)  $E$ : “at least one head