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SOME APPLICATIONS OF FRACTIONAL CALCULUS OPERATORS TO CERTAIN SUBCLASSES OF UNIVALENT FUNCTIONS

BY

M. K. AOUF, A. SHAMANDY AND M. F. YASSEN

Abstract. The object of the present paper is to prove distortion theorems for certain fractional integral operator of functions in the subclasses $S_0(\alpha,\beta)$ and $C_0(\alpha,\beta)$ of analytic and univalent functions in the unit disc U.

1. Introduction

Let T denote the class of functions of the form

$$f(z)=a_1z-\sum_{n=2}^{\infty}a_nz^n \quad (a_1>0;\ a_n\geq 0), \tag{1.1}$$

which are analytic in the unit disc $U=\{z:|z|<1\}$. A function $f(z)\in T$ is said to be in the class $S_0(\alpha,\beta)$ if it satisfies the following condition

$$\left|\frac{\frac{zf'(z)}{f(z)}-1}{\alpha\frac{zf'(z)}{f(z)}+1}\right|<\beta \tag{1.2}$$

for some $0\leq\alpha<1$, $0<\beta\leq 1$ and for all $z\in U$. Also let $C_0(\alpha,\beta)$ denote the class of functions $f(z)\in T$ such that $zf'(z)\in S_0(\alpha,\beta)$. The classes $S_0(\alpha,\beta)$ and $C_0(\alpha,\beta)$ were studied by Owa [1].

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In order to prove our results for functions belonging to the classes $S_0(\alpha,\beta)$ and $C_0(\alpha,\beta)$, we shall require the following lemmas given by Owa [1].

Lemma 1. *Let the function $f(z)$ be defined by* (1.1). *Then $f(z) \in S_0(\alpha,\beta)$ if and only if*

$$\sum_{n=2}^{\infty}\{(n-1)+\beta(1+\alpha n)\}a_n \leq \beta(1+\alpha)a_1. \tag{1.3}$$

The result is sharp.

Lemma 2. *Let the function $f(z)$ be defined by* (1.1). *Then $f(z) \in C_0(\alpha,\beta)$ if and only if*

$$\sum_{n=2}^{\infty} n\{(n-1)+\beta(1+\alpha n)\}a_n \leq \beta(1+\alpha)a_1. \tag{1.4}$$

The result is sharp.

2. Fractional Integral Operator

We need the following definition of fractional integral operator given by Srivastava, Saigo and Owa [4].

Definition 1. For real numbers $\rho > 0$, γ and δ, the fractional integral operator $I_{0,z}^{\rho,\gamma,\delta}$ is defined by

$$I_{0,z}^{\rho,\gamma,\delta} f(z) = \frac{z^{-\rho-\gamma}}{\Gamma(\rho)} \int_0^z (z-t)^{\rho-1} F(\rho+\gamma, -\delta; \rho; 1-\frac{t}{z}) f(t)dt \tag{2.1}$$

where $f(z)$ is an analytic function in a simply connected region of the z-plane containing the origin with the order

$$f(z) = O(|z|^{\epsilon}), z \longrightarrow 0,$$

where

$$\begin{aligned} &\epsilon > \max(0, \gamma-\delta) - 1, \\ &F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n, \end{aligned} \tag{2.2}$$

$(\nu)_n$ is the Pochhammer symbol defined by

$$(\nu)_n = \frac{\Gamma(\nu+n)}{\Gamma(\nu)} = \begin{cases} 1 & (n=0) \\ \nu(\nu+1)\dots(\nu+n-1) & (n \in N = \{1,2,\dots\}) \end{cases} \tag{2.3}$$

and the multiplicity of $(z-t)^{\rho-1}$ is removed by requiring log $(z-t)$ to be real when $z-t>0$.

Remark. For $\gamma = -\rho$, we note that

$$I_{0,z}^{\rho,-\rho,\delta} f(z) = D_z^{-\rho} f(z), \tag{2.4}$$

where $D_z^{-\rho} f(z)$ is the fractional integral of order ρ of $f(z)$ which was introduced by Owa ([2],[3]).

In order to prove our results for the fractional integral operators, we have to recall here the following lemma due to Srivastava, Saigo and Owa [4].

Lemma 3. *If $\rho > 0$ and $n > \gamma - \delta - 1$, then*

$$I_{0,z}^{\rho,\gamma,\delta} z^n = \frac{\Gamma(n+1)\Gamma(n-\gamma+\delta+1)}{\Gamma(n-\gamma+1)\Gamma(n+\rho+\delta+1)} z^{n-\gamma}. \tag{2.5}$$

With the aid of lemma 3, we prove

Theorem 1. *Let $\rho > 0$, $\gamma < 2$, $\rho + \delta > -2$, $\gamma - \delta < 2$, $\gamma(\rho+\delta) \leq 3\rho$. If the function $f(z)$ defined by (1.1) is in the class $S_0(\alpha, \beta)$, then*

$$\left| I_{0,z}^{\rho,\gamma,\delta} f(z) \right| \geq \frac{a_1 \Gamma(2-\gamma+\delta)|z|^{1-\gamma}}{\Gamma(2-\gamma)\Gamma(2+\rho+\delta)} \left\{ 1 - \frac{2\beta(1+\alpha)(2-\gamma+\delta)}{\{1+\beta(1+2\alpha)\}(2-\gamma)(2+\rho+\delta)} |z| \right\} \tag{2.6}$$

and

$$\left| I_{0,z}^{\rho,\gamma,\delta} f(z) \right| \leq \frac{a_1 \Gamma(2-\gamma+\delta)|z|^{1-\gamma}}{\Gamma(2-\gamma)\Gamma(2+\rho+\delta)} \left\{ 1 + \frac{2\beta(1+\alpha)(2-\gamma+\delta)}{\{1+\beta(1+2\alpha)\}(2-\gamma)(2+\rho+\delta)} |z| \right\} \tag{2.7}$$

for $z \in U_0$,where

$$U_0 = \begin{cases} U & (\gamma \leq 1) \\ U - \{0\} & (\gamma > 1). \end{cases}$$

The equalities in (2.6) *and* (2.7) *are attained by the function* $f(z)$ *given by*

$$f(z) = a_1 z - \frac{\beta(1+\alpha)a_1}{\{1+\beta(1+2\alpha)\}} z^2. \tag{2.8}$$

Proof. By using Lemma 3, we have

$$\begin{aligned} I_{0,z}^{\rho,\gamma,\delta} f(z) =& \frac{a_1 \Gamma(2-\gamma+\delta)}{\Gamma(2-\gamma)\Gamma(2+\rho+\delta)} z^{1-\gamma} \\ & - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(n-\gamma+\delta+1)}{\Gamma(n-\gamma+1)\Gamma(n+\rho+\delta+1)} a_n z^{n-\gamma}. \end{aligned} \tag{2.9}$$

Letting

$$\begin{aligned} H(z) =& \frac{\Gamma(2-\gamma)\Gamma(2+\rho+\delta)}{\Gamma(2-\gamma+\delta)} z^{\gamma} I_{0,z}^{\rho,\gamma,\delta} f(z) \\ =& a_1 z - \sum_{n=2}^{\infty} h(n) a_n z^n, \end{aligned} \tag{2.10}$$

where

$$h(n) = \frac{(2-\gamma+\delta)_{n-1}(1)_n}{(2-\gamma)_{n-1}(2+\rho+\delta)_{n-1}} \qquad (n \geq 2), \tag{2.11}$$

we can see that $h(n)$ is non-increasing for integers n $(n \geq 2)$, and we have

$$0 < h(n) \leq h(2) = \frac{2(2-\gamma+\delta)}{(2-\gamma)(2+\rho+\delta)}, \tag{2.12}$$

from Lemma 1, we have

$$\sum_{n=2}^{\infty} a_n \leq \frac{\beta(1+\alpha)a_1}{\{1+\beta(1+2\alpha)\}}. \tag{2.13}$$

Therefore, by using (2.12) amd (2.13),we have

$$\begin{aligned} |H(z)| \geq& a_1|z| - h(2)|z|^2 \sum_{n=2}^{\infty} a_n \\ \geq& a_1|z| - \frac{2\beta(1+\alpha)(2-\gamma+\delta)a_1}{\{1+\beta(1+2\alpha)\}(2-\gamma)(2+\rho+\delta)} |z|^2 \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} |H(z)| \leq& a_1|z| + h(2)|z|^2 \sum_{n=2}^{\infty} a_n \\ \leq& a_1|z| + \frac{2\beta(1+\alpha)(2-\gamma+\delta)a_1}{\{1+\beta(1+2\alpha)\}(2-\gamma)(2+\rho+\delta)} |z|^2. \end{aligned} \tag{2.15}$$

This completes the proof of Theorem 1.

Similarly, by applying lemma 2 (instead of Lemma 1) to the functions $f(z)$ belonging to the class $C_0(\alpha,\beta)$, we can derive the following theorem.

Theorem 2. *Let $\rho > 0$, $\gamma < 2$, $\rho+\delta > -2$, $\gamma-\delta < 2$, $\gamma(\rho+\delta) \leq 3\rho$. If the function $f(z)$ defined by* (1.1) *is in the class $C_0(\alpha,\beta)$, then*

$$|I_{0,z}^{\rho,\gamma,\delta} f(z)| \geq \frac{a_1\Gamma(2-\gamma+\delta)|z|^{1-\gamma}}{\Gamma(2-\gamma)\Gamma(2+\rho+\delta)}\left\{1-\frac{\beta(1+\alpha)(2-\gamma+\delta)}{\{1+\beta(1+2\alpha)\}(2-\gamma)(2+\rho+\delta)}|z|\right\} \quad (2.16)$$

$$|I_{0,z}^{\rho,\gamma,\delta} f(z)| \leq \frac{a_1\Gamma(2-\gamma+\delta)|z|^{1-\gamma}}{\Gamma(2-\gamma)\Gamma(2+\rho+\delta)}\left\{1+\frac{\beta(1+\alpha)(2-\gamma+\delta)}{\{1+\beta(1+2\alpha)\}(2-\gamma)(2+\rho+\delta)}|z|\right\} \quad (2.17)$$

for $z \in U_0$, where U_0 is defined in Theorem 1. *The equalites in* (2.16) *and* (2.17) *are attained by the function $f(z)$ given by*

$$f(z) = a_1 z - \frac{\beta(1+\alpha)a_1}{2\{1+\beta(1+2\alpha)\}}z^2. \quad (2.18)$$

Taking $\gamma = -\rho = -k$ in Theorems 1 *and* 2, *we get*

Corollary 1. *Let the function $f(z)$ defined by* (1.1) *be in the calss $S_0(\alpha,\beta)$. Then we have*

$$|D_z^{-k} f(z)| \geq \frac{a_1|z|^{1+k}}{\Gamma(2+k)}\left\{1-\frac{2\beta(1+\alpha)}{\{1+\beta(1+2\alpha)\}(2+k)}|z|\right\} \quad (2.19)$$

and

$$|D_z^{-k} f(z)| \leq \frac{a_1|z|^{1+k}}{\Gamma(2+k)}\left\{1+\frac{2\beta(1+\alpha)}{\{1+\beta(1+2\alpha)\}(2+k)}|z|\right\} \quad (2.20)$$

for $k > 0$ and $z \in U$. The result is sharp for the function $f(z)$ defined by (2.8).

Corollary 2. *Let the function $f(z)$ defined by* (1.1) *be in the class $C_0(\alpha,\beta)$. Then we have*

$$|D_z^{-k} f(z)| \geq \frac{a_1|z|^{1+k}}{\Gamma(2+k)}\left\{1-\frac{\beta(1+\alpha)}{\{1+\beta(1+2\alpha)\}(2+k)}|z|\right\} \quad (2.21)$$

and

$$|D_z^{-k} f(z)| \leq \frac{a_1|z|^{1+k}}{\Gamma(2+k)} \left\{ 1 + \frac{\beta(1+\alpha)}{\{1+\beta(1+2\alpha)\}(2+k)} |z| \right\} \tag{2.22}$$

for $k > 0$ *and* $z \in U$. *The result is sharp for the function* $f(z)$ *defined by* (2.18).

Remark 1. Taking $k = 0$ in Corollaries 1 and 2, we get the resutls of Owa [1].

References

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Department of Mathematics, Faculty of science, University of Mansoura, Mansoura, Egypt.