Probability and Its Applications

Rolf Schneider
Wolfgang Weil

Stochastic and Integral Geometry

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Preface

Stochastic Geometry deals with mathematical models for random geometric structures and spatial data, as they frequently arise in modern applications. As a mathematical discipline, stochastic geometry came into life in the last third of the twentieth century, but its roots and the close connections between geometric probability and integration techniques using invariant measures (though not under this name) date back much farther. The famous Buffon needle problem of 1777 was solved by what seems to be the first application of integral calculus to a probability question. A variety of problems in *Geometric Probability* was treated in the late nineteenth and early twentieth century. After the role of invariant measures had become clear, the discipline of *Integral Geometry* was initiated in the 1930s, mostly by Wilhelm Blaschke and his school. The book *Integral Geometry and Geometric Probability* by Luis Santaló (1976) summarizes the concepts and results of the preceding development. Interpretations of integral geometric results in terms of geometric probability abound in that work. At that time, David Kendall and Georges Matheron had already developed, independently, a theory of *Random Sets*, and Roger Miles had written his pioneering thesis on Poisson processes of certain geometric objects. The book *Random Sets and Integral Geometry* by Matheron (1975) presented the new field of Stochastic Geometry in its intimate relation with Integral Geometry. Applications in *Spatial Statistics* and *Stereology*, later also in *Image Analysis*, contributed to a rapid development. The classical integral geometry of Euclidean spaces is well suited to the treatment of random sets and point processes with invariance properties, like stationarity and isotropy. The necessity of studying structures which exhibit anisotropy, or even without spatial homogeneity, grew hand in hand with new developments in integral geometry, coming from *Geometric Measure Theory*. In particular, Federer's local formulas for curvature measures proved useful, and *Translative Integral Geometry* was promoted, meeting the needs of stationary structures.

Over many years, we both gave courses on Integral Geometry or Stochastic Geometry in Freiburg and Karlsruhe. This led to the joint publication of lecture notes in German, under the titles of *Integralgeometrie* (1992) and

Stochastische Geometrie (2000). It was always our plan later to amalgamate both topics in one extended monograph in English. During the time we worked on this project, the field of stochastic geometry has expanded considerably in various directions, too many to include them all in one volume. We decided to concentrate on our original idea, namely to present the basic models of stochastic geometry and their properties, the fundamental concepts and formulas of integral geometry, and the interrelations between these two fields.

In this book, therefore, we have three main aims: to give a sound mathematical foundation for the most basic and general models of stochastic geometry, namely random closed sets, particle processes, and random mosaics, to introduce the reader to the parts of integral geometry that are relevant for the applications in stochastic geometry, and, naturally, to demonstrate such applications. Since the strength of integral geometry lies in the computation of mean values and in integral transformations, this means that we develop mainly a 'first order theory' of stochastic geometry, centering around expectations. This restricted concept, with its foundational character, implies that essential and interesting parts of stochastic geometry are missing: we do not treat special point process models other than Poisson processes, nor higher order moment measures, limit theorems, spatial statistics, practical procedures, simulations; however, we comment on some of these developments in the section notes. The integral geometry here is taylored to its use in stochastic geometry; this influences the selection of topics as well as the approach, which is measure theoretic rather than differential geometric. Another restriction may be seen in the predominance of *invariance* and *independence*. The first means that we study (except in one chapter providing an outlook) only random sets and geometric point processes that are stationary (spatially homogeneous) or even stationary and isotropic, in distribution. Invariance of measures and distributions is the leitmotiv of this volume; it underlies both the stochastic geometry parts and the integral geometric parts. On the stochastic side, there is a preference for independence assumptions, as for example in the prominent role of Poisson processes, with their strong independence properties. Very often, only invariance and independence assumptions allow simple approaches and lead to beautiful results. The confinement to the fundamentals of stochastic geometry leaves us room for emphasizing the geometry; in fact, in integral as well as in stochastic geometry, we draw a richer picture than sketched above, and we include various topics of geometric appeal. For example, there is a chapter on *Geometric Probability*, since this area has seen a recent revival with many interesting problems and results.

Naturally, this book employs notions and results from other fields. We make use of some basic facts from general topology, from the theory of topological groups and homogeneous spaces of Euclidean geometry and their invariant measures, and from the geometry of convex sets; further, some more specialized results concerning geometric inequalities and additive functionals on convex bodies are needed. Anticipating that the familiarity of the readers with these topics will not be uniform, we have collected the required material in an Appendix; this should be consulted whenever necessary. This also allows us to start directly with the fundamental notion in this book, the concept of a random closed set.

We are grateful to many colleagues for their helpful comments on early drafts of our book. Special thanks go to Paul Goodey, Günter Last and Werner Nagel, for providing useful hints after reading parts of the final manuscript, and in particular to Daniel Hug, who has carefully read all of it. He prevented us from including a number of flaws and made many suggestions for improvements. We also thank the Mathematisches Forschungsinstitut Oberwolfach for giving us the opportunity to spend some time, working on our mansucript, in their wonderful 'Research in Pairs' programme.

Freiburg i. Br., Karlsruhe *Rolf Schneider*
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Contents

Part II Integral Geometry

Part III Selected Topics from Stochastic Geometry

1

Prolog

1.1 Introduction

Since this book is about relations between stochastic geometry and integral geometry, we begin with an imaginary experiment that demonstrates the need for and use of integral geometry for certain geometric probability questions and at the same time leads in a natural way to a basic model of stochastic geometry.

We assume that K and W are given convex bodies (nonempty compact convex sets) in d-dimensional Euclidean space $\mathbb{R}^d$. The body K serves to generate a random field of congruent copies of K, and the body W plays the role of an 'observation window'. The random field consists of countably many congruent copies of K which are laid out in space randomly and independently, overlappings being allowed. The number of bodies in the random field that hit (that is, have nonempty intersection with) the observation window W is a random variable. We ask for its distribution. This is, of course, not a meaningful question, as long as no stochastic model for the random field of convex bodies is specified. In a few steps, we shall introduce some natural assumptions, which motivate a precise model and lead to an explicit formula for the desired distribution.

In the first step, we consider a much simpler situation. We take a ball B_r of radius r and origin 0 that contains the observation window W, and we consider only one randomly moving copy of K, under the condition that it hits B_r. We ask for the probability that it also hits W. There is a geometrically very natural way of specifying a probability distribution of a randomly moving convex body that satisfies the side condition. A random congruent copy of K can be represented in the form $\widetilde{g}K$, where $\widetilde{g}$ is a random element of the group G_d of rigid motions. The locally compact group G_d carries an essentially unique Haar measure, that is, a locally finite Borel measure that is similarly under left and right multiplications and is not identically zero. We denote this measure, with a suitable normalization, by μ. A natural probability distribution of a random congruent copy of K hitting B_r is then obtained