SOOCHOW JOURNAL OF MATHEMATICS
Volume 21, No. 3, pp. 299-311, July 1995

ON SUBCLASSES OF UNIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS .IV

BY

M. K. AOUF, A. SHAMANDY AND M. F. YASSEN

Abstract. The object of the present paper is to obtain closure theorems and integral operators of functions in the classes $S^*(\alpha,\beta,\mu)$ and $C^*(\alpha,\beta,\mu)$ $(0 \le \alpha < 1,\ 0 < \beta \le 1,\ 0 \le \mu \le 1)$ consisting of analytic and univalent functions with negative coefficients. Furthermore, some interesting distortion inequalities for certain fractional integral operator are shown.

1. Introduction

Let S denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic and univalent in the unit disc $U = \{z : |z| < 1\}$. Let T be the subclass of S consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \ge 0). \tag{1.2}$$

A function $f(z) \in T$ is said to be in the class $S^*(\alpha,\beta,\mu)$ if and only if

$$\left| \frac{\dfrac{zf'(z)}{f(z)} - 1}{\mu \dfrac{zf'(z)}{f(z)} + 1 - (1+\mu)\alpha} \right| < \beta \tag{1.3}$$

Received July 6, 1994.
AMS Subject Classification. 30C45.
Keywords▷ analytic, univalent, integral operator.

for some α $(0 \le \alpha < 1)$, β $(0 < \beta \le 1)$, μ $(0 \le \mu \le 1)$, and for all $z \in U$. Further $f(z) \in T$ is said to be in the class $C^*(\alpha, \beta, \mu)$ if and only if $zf'(z) \in S^*(\alpha, \beta, \mu)$. The classes $S^*(\alpha, \beta, \mu)$ and $C^*(\alpha, \beta, \mu)$ were studied by Owa and Aouf [7] and Aouf [1].

We note that:

(i) $S^*(\alpha, \beta, 1) = S^*(\alpha, \beta)$ and $C^*(\alpha, \beta, 1) = C^*(\alpha, \beta)$ were studied by Gupta and Jain [2], Owa [6] and Kumar and Shukla [3].

(ii) $S^*(\alpha, 1, 1) = S^*(\alpha)$ and $C^*(\alpha, 1, 1) = C^*(\alpha)$ were studied by Silverman [8].

In order to show our results, we need the following lemmas given by Owa and Aouf [7].

Lemma 1. *A function $f(z)$ defined by* (1.2) *is in the class $S^*(\alpha, \beta, \mu)$ if and only if*

$$\sum_{n=2}^{\infty} D(n, \alpha, \beta, \mu) a_n \le (1+\mu)\beta(1-\alpha), \tag{1.4}$$

where

$$D(n, \alpha, \beta, \mu) = (n-1) + \beta[\mu n + 1 - (1+\mu)\alpha]. \tag{1.5}$$

The result is sharp.

Lemma 2. *A function $f(z)$ defined by* (1.2) *is in the class $C^*(\alpha, \beta, \mu)$ if and only if*

$$\sum_{n=2}^{\infty} n D(n, \alpha, \beta, \mu) a_n \le (1+\mu)\beta(1-\alpha). \tag{1.6}$$

The result is sharp.

2. Closure Theorems

Let the functions $f_j(z)$ be defined, for $j = 1, 2, \dots, m$, by

$$f_j(z) = z - \sum_{n=2}^{\infty} a_{n,j} z^n \quad (a_{n,j} \ge 0) \tag{2.1}$$

for $z \in U$.

We shall prove the following results for the closure of functions in the classes $S^*(\alpha,\beta,\mu)$ and $C^*(\alpha,\beta,\mu)$.

Theorem 1. *Let the functions* $f_j(z)$ $(j=1,2,\dots,m)$ *defined by* (2.1) *be in the class* $S^*(\alpha,\beta,\mu)$. *Then the function* $h(z)$ *defined by*

$$h(z)=z-\sum_{n=2}^{\infty}b_nz^n \tag{2.2}$$

also belongs to the calss $S^*(\alpha,\beta,\mu)$, *where*

$$b_n=\frac{1}{m}\sum_{j=1}^{m}a_{n,j}. \tag{2.3}$$

Proof. Since $f_j(z)\in S^*(\alpha,\beta,\mu)$, it follows from Lemma 1 that

$$\sum_{n=2}^{\infty}D(n,\alpha,\beta,\mu)a_{n,j}\le(1+\mu)\beta(1-\alpha),\quad j=1,2,\dots,m.$$

Therefore

$$\begin{aligned}\sum_{n=2}^{\infty}D(n,\alpha,\beta,\mu)b_n&=\sum_{n=2}^{\infty}D(n,\alpha,\beta,\mu)\left\{\frac{1}{m}\sum_{j=1}^{m}a_{n,j}\right\}\\&\le(1+\mu)\beta(1-\alpha).\end{aligned} \tag{2.4}$$

Hence by Lemma 1, $h(z)\in S^*(\alpha,\beta,\mu)$. Thus we have the theorem.

By using Lemma 2, we have

Theorem 2. *Let the functions* $f_j(z)$ $(j=1,2,\dots,m)$ *defined by* (2.1) *be in the class* $C^*(\alpha,\beta,\mu)$. *Then the function* $h(z)$ *defined by* (2.2) *also belongs to the class* $C^*(\alpha,\beta,\mu)$ *under the condition* (2.3).

Theorem 3. *Let the functions* $f_j(z)$ $(j=1,2,\dots,m)$ *defined by* (2.1) *be in the class* $S^*(\alpha,\beta,\mu)$. *Then the function* $h(z)$ *defined by*

$$h(z)=\sum_{j=1}^{m}d_jf_j(z)\quad(d_j\ge 0) \tag{2.5}$$

is also in the same class $S^*(\alpha, \beta, \mu)$, *where*

$$\sum_{j=1}^{m} d_j = 1. \tag{2.6}$$

Proof. According to the definition of $h(z)$, we can write that

$$h(z) = z - \sum_{n=2}^{\infty} \left[\sum_{j=1}^{m} d_j a_{n,j}\right] z^n. \tag{2.7}$$

By means of Lemma 1, we have

$$\sum_{n=2}^{\infty} D(n, \alpha, \beta, \mu) a_{n,j} \leq (1+\mu)\beta(1-\alpha) \tag{2.8}$$

for every $j = 1, 2, \ldots, m$. Hence we can observe that

$$\sum_{n=2}^{\infty} D(n, \alpha, \beta, \mu) \left[\sum_{j=1}^{m} d_j a_{n,j}\right] = \sum_{j=1}^{m} d_j \left[\sum_{n=2}^{\infty} D(n, \alpha, \beta, \mu) a_{n,j}\right]$$
$$\leq \left[\sum_{j=1}^{m} d_j\right] (1+\mu)\beta(1-\alpha) = (1+\mu)\beta(1-\alpha) \tag{2.9}$$

which implies that $h(z) \in S^*(\alpha, \beta, \mu)$. Thus we have the theorem.

By using Lemma 2, we have

Theorem 4. *Let the functions* $f(z)$ *defined by* (2.1) *be in the class* $C^*(\alpha, \beta, \mu)$ *for every* $j = 1, 2, \ldots, m$. *Then the functin* $h(z)$ *defined by* (2.5) *is also belongs to the same class* $C^*(\alpha, \beta, \mu)$ *under the condition* (2.6).

Theorem 5. *Let the function* $f_1(z)$ *defined by* (2.1) *be in the class* $S^*(\alpha, \beta, \mu)$ *and the function* $f_2(z)$ *defined by* (2.1) *be in the class* $C^*(\alpha, \beta, \mu)$. *Then the function* $k(z)$ *defined by*

$$k(z) = z - \frac{2}{3} \sum_{n=2}^{\infty} (a_{n,1} + a_{n,2}) z^n \tag{2.10}$$

is in the class $S^*(\alpha, \beta, \mu)$.

Proof. Since $f_1(z) \in S^*(\alpha,\beta,\mu)$ and $f_2(z) \in C^*(\alpha,\beta,\mu)$, by using Lemmas 1 and 2 we get, respectively,

$$\sum_{n=2}^{\infty} D(n,\alpha,\beta,\mu)a_{n,1} \leq (1+\mu)\beta(1-\alpha) \tag{2.11}$$

and

$$\sum_{n=2}^{\infty} D(n,\alpha,\beta,\mu)a_{n,2} \leq \frac{(1+\mu)\beta(1-\alpha)}{2}. \tag{2.12}$$

Therefore, we have

$$\frac{2}{3}\sum_{n=2}^{\infty} D(n,\alpha,\beta,\mu)(a_{n,1}+a_{n,2}) \leq (1+\mu)\beta(1-\alpha) \tag{2.13}$$

which implies that $k(z) \in S^*(\alpha,\beta,\mu)$, and the proof of Theorem 5 is thus completed.

3. Integral Operators

Theorem 6. *Let the function $f(z)$ defined by (1.2) be in the class $S^*(\alpha,\beta,\mu)$ and let c be a real number such that $c > -1$. Then the function $F(z)$ defined by*

$$F(z) = \frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt \tag{3.1}$$

also belongs to the class $S^*(\alpha,\beta,\mu)$.

Proof. From the representation of $F(z)$, it follows that

$$F(z) = z - \sum_{n=2}^{\infty} b_n z^n, \tag{3.2}$$

where

$$b_n = \left(\frac{c+1}{c+n}\right)a_n. \tag{3.3}$$

Therefore,

$$\sum_{n=2}^{\infty} D(n,\alpha,\beta,\mu)b_n = \sum_{n=2}^{\infty} D(n,\alpha,\beta,\mu)\left(\frac{c+1}{c+n}\right)a_n$$

$$\leq \sum_{n=2}^{\infty} D(n,\alpha,\beta,\mu)a_n \leq (1+\mu)\beta(1-\alpha), \tag{3.4}$$

since $f(z) \in S^*(\alpha,\beta,\mu)$. Hence, by Lemma 1, $F(z) \in S^*(\alpha,\beta,\mu)$.

Theorem 7. *Let c be a real number such that $c > -1$. If $F(z) \in S^*(\alpha,\beta,\mu)$, then the function $f(z)$ defined by (3.1) is univalent in $|z| < R^*$, where*

$$R^* = \inf_n \left(\frac{D(n,\alpha,\beta,\mu)(c+1)}{n(1+\mu)\beta(1-\alpha)(c+n)} \right)^{\frac{1}{n-1}} \quad (n \geq 2). \tag{3.5}$$

The result is sharp.

Proof. Let $F(z) = z - \sum_{n=2}^{\infty} a_n z^n$ $(a_n \geq 0)$. It follows from (3.1) that

$$\begin{aligned} f(z) =& \frac{z^{1-c}[z^c F(z)]'}{(c+1)} \quad (c > -1) \\ =& z - \sum_{n=2}^{\infty} \left(\frac{c+n}{c+1} \right) a_n z^n. \end{aligned} \tag{3.6}$$

In order to obtain the required result it suffices to show that $|f'(z) - 1| < 1$ in $|z| < R^*$.

Now $|f'(z) - 1| < 1$ if

$$\sum_{n=2}^{\infty} \frac{n(c+n)}{(c+1)} a_n |z|^{n-1} < 1. \tag{3.7}$$

According to Lemma 1, we have

$$\sum_{n=2}^{\infty} \frac{D(n,\alpha,\beta,\mu)}{(1+\mu)\beta(1-\alpha)} a_n \leq 1. \tag{3.8}$$

Hence (3.7) will be true if

$$\frac{n(c+n)|z|^{n-1}}{(c+1)} < \frac{D(n,\alpha,\beta,\mu)}{(1+\mu)\beta(1-\alpha)}$$

or if

$$|z| < \left(\frac{D(n,\alpha,\beta,\mu)(c+1)}{n(1+\mu)\beta(1-\alpha)(c+n)} \right)^{\frac{1}{n-1}} \quad (n \geq 2). \tag{3.9}$$

Therefore $f(z)$ is univalent in $|z| < R^*$. Sharpness follows if we take

$$f(z) = z - \frac{(1+\mu)\beta(1-\alpha)(c+n)}{D(n,\alpha,\beta,\mu)(c+1)} z^n \quad (n \geq 2). \tag{3.10}$$

Theorem 8. *Let c be a real number such that $c > -1$. If $F(z) = z - \sum_{n=2}^{\infty} a_n z^n$ $(a_n \geq 0)$ belongs to the class $S^*(\alpha, \beta, \mu)$, then the function $f(z)$ defined by* (3.1) *is starlike of order ρ $(0 \leq \rho < 1)$ in $|z| < r^*(\rho, \alpha, \beta, \mu)$, where*

$$r^*(\rho, \alpha, \beta, \mu) = \inf_n \left[\left(\frac{1-\rho}{n-\rho} \right) \left(\frac{c+1}{c+n} \right) \frac{D(n, \alpha, \beta, \mu)}{(1+\mu)\beta(1-\alpha)} \right]^{\frac{1}{n-1}} \quad (n \geq 2). \tag{3.11}$$

The result is sharp.

Proof. In order to establish the required result it suffices to show that

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < (1-\rho) \quad \text{in} \quad |z| < r^*(\rho, \alpha, \beta, \mu).$$

Now

$$\begin{aligned} \left| \frac{zf'(z)}{f(z)} - 1 \right| &= \left| \frac{-\sum_{n=2}^{\infty} (n-1) \left(\frac{c+n}{c+1} \right) a_n z^{n-1}}{1 - \sum_{n=2}^{\infty} \left(\frac{c+n}{c+1} \right) a_n z^{n-1}} \right| \\ &\leq \frac{\sum_{n=2}^{\infty} (n-1) \left(\frac{c+n}{c+1} \right) a_n |z|^{n-1}}{1 - \sum_{n=2}^{\infty} \left(\frac{c+n}{c+1} \right) a_n |z|^{n-1}} \\ &< (1-\rho) \end{aligned} \tag{3.12}$$

provided

$$\sum_{n=2}^{\infty} \left(\frac{n-\rho}{1-\rho} \right) \left(\frac{c+n}{c+1} \right) a_n |z|^{n-1} < 1. \tag{3.13}$$

By using (3.8), the inequality (3.13) holds if

$$\left(\frac{n-\rho}{1-\rho} \right) \left(\frac{c+n}{c+1} \right) |z|^{n-1} < \frac{D(n, \alpha, \beta, \mu)}{(1+\mu)\beta(1-\alpha)} \quad (n \geq 2)$$

or if

$$|z| < \left[\left(\frac{1-\rho}{n-\rho} \right) \left(\frac{c+1}{c+n} \right) \frac{D(n, \alpha, \beta, \mu)}{(1+\mu)\beta(1-\alpha)} \right]^{\frac{1}{n-1}} \quad (n \geq 2).$$

Hence, $f(z) \in S^*(\rho)$ in $|z| < r^*(\rho, \alpha, \beta, \mu)$. Sharpness follows if we take the function $F(z)$ given by

$$F(z) = z - \frac{(1+\mu)\beta(1-\alpha)}{D(n, \alpha, \beta, \mu)} z^n \quad (n \geq 2). \tag{3.14}$$

Remark. Putting $c = \mu = 1$ in Theorem 8, we get the result of Kumar and Shukla [3, Theorem 2].

Theorem 9. *Let the function $f(z)$ be defined by (1.2). If $f(z) \in S^*(\alpha, \beta, \mu)$, then the function $F(z)$ defined by (3.1) belongs to $S^*(\rho)$, where*

$$\rho = \frac{(c+2) + \beta[(2\mu - c) + c(1+\mu)\alpha]}{(c+2) + \beta[(3+c)\mu + 1 - (1+\mu)\alpha]}. \tag{3.15}$$

The result is sharp. Futher, the converse need not be true.

Proof. Let $F(z) = z - \sum_{n=2}^{\infty} b_n z^n \in S^*(\sigma)$, where b_n is given by (3.3), then, by Lemma 1, it holds if and only if

$$\sum_{n=2}^{\infty} \left(\frac{n-\sigma}{1-\sigma}\right) b_n \leq 1. \tag{3.16}$$

Thus we have to find the largest value of σ so that the inequality (3.16) holds. Now by using (3.8), (3.16) holds if

$$\left(\frac{n-\sigma}{1-\sigma}\right) b_n \leq \frac{D(n,\alpha,\beta,\mu)}{(1+\mu)\beta(1-\alpha)} a_n \quad (n \geq 2)$$

or if

$$\left(\frac{n-\sigma}{1-\sigma}\right)\left(\frac{c+1}{c+n}\right) \leq \frac{D(n,\alpha,\beta,\mu)}{(1+\mu)\beta(1-\alpha)} \quad (n \geq 2), \tag{3.17}$$

which is equivalent to

$$\begin{aligned} \sigma \leq & \frac{(c+n)D(n,\alpha,\beta,\mu) - (c+1)n(1+\mu)\beta(1-\alpha)}{(c+n)D(n,\alpha,\beta,\mu) - (c+1)(1+\mu)\beta(1-\alpha)} \\ =& \rho_n, \text{ say}, (n \geq 2). \end{aligned} \tag{3.18}$$

It is easy to verify that ρ_n is an increasing function of n $(n \geq 2)$. Therefore $\rho = \inf_{n\geq 2} \rho_n = \rho_2$ and, hence

$$\rho = \frac{(c+2) + \beta[(2\mu - c) + c(1+\mu)\alpha]}{(c+2) + \beta[(3+c)\mu + 1 - (1+\mu)\alpha]}.$$

To show the sharpness we take the function $f(z)$ given by

$$f(z) = z - \frac{(1+\mu)\beta(1-\alpha)}{D(2,\alpha,\beta,\mu)} z^2. \tag{3.19}$$

Then

$$F(z) = z - \frac{(c+1)(1+\mu)\beta(1-\alpha)}{(c+2)D(2,\alpha,\beta,\mu)} z^2, \tag{3.20}$$

and, therefore

$$\begin{aligned}\frac{zF'(z)}{F(z)} &= \frac{(c+2)D(2,\alpha,\beta,\mu) - 2(c+1)(1+\mu)\beta(1-\alpha)z}{(c+2)D(2,\alpha,\beta,\mu) - (c+1)(1+\mu)\beta(1-\alpha)z} \\ &= \frac{(c+2) + \beta[(2\mu - c) + c(1+\mu)\alpha]}{(c+2) + \beta[(3+c)\mu + 1 - (1+\mu)\alpha]}, \text{ for } z = 1.\end{aligned}$$

Hence, the result is sharp.

We now show that the converse of the theorem need not be true. To this end we consider the function

$$F(z) = z - \left(\frac{1-\rho}{3-\rho}\right) z^3. \tag{3.21}$$

Lemma 1 guarantees that $F(z) \in S^*(\rho)$. But the corresponding function

$$f(z) = z - \frac{(c+3)(1-\rho)}{(c+1)(3-\rho)} z^3 \tag{3.22}$$

does not belong to $S^*(\alpha,\beta,\mu)$, since, for this $f(z)$ the coefficient inequality of Lemma 1 is not satisfied.

Corollary 1. *Let the function $f(z)$ be defined by (1.2). If $f(z) \in S^*(\alpha)$ $(0 \le \alpha < 1)$, then the function $F(z)$ defined by (3.1) belongs to the class $S^*(\frac{2+c\alpha}{c+3-\alpha})$. The result is sharp. The converse need not be true.*

Remark.

(1) Putting $c = \mu = 1$ in Theorem 9, we get the result of Kumar and Shukla [3, Theorem 1].

(2) Putting $\alpha = 0$ and $c = 1$ in Corollary 1, we get the result of Kumar and Shukla [3, Corollary 1].

4. Fractional Integral Operator

We need the following definition of fractional integral operator given by Srivastava, Saigo and Owa [9].

Definition 1. For real numbers $\rho > 0$, δ and η, the fractional integral operator $I_{0,z}^{\rho,\delta,\eta}$ is defined by

$$I_{0,z}^{\rho,\delta,\eta} f(z) = \frac{z^{-\rho-\delta}}{\Gamma(\rho)} \int_0^z (z-t)^{\rho-1} F(\rho+\delta, -\eta; \rho; 1-\frac{t}{z}) f(t) dt \tag{4.1}$$

where $f(z)$ is an analytic functin in a simply connected region of the z-plane containing the origin with the order

$$f(z) = O(|z|^{\epsilon}), \quad z \to 0,$$

where $\epsilon > \text{Max}(0, \delta - \eta) - 1$,

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n, \tag{4.2}$$

where $(\nu)_n$ is the Pochhammer symbol defined by

$$(\nu)_n = \frac{\Gamma(\nu+n)}{\Gamma(\nu)} = \begin{cases} 1 & (n=0), \\ \nu(\nu+1)\cdots(\nu+n-1) & (n \in N = \{1, 2, \ldots\}), \end{cases} \tag{4.3}$$

and the multiplicity of $(z-t)^{\rho-1}$ is removed by requiring $\log(z-t)$ to be real when $z - t > 0$.

Remark. For $\delta = -\rho$, we note that

$$I_{0,z}^{\rho,-\rho,\eta} f(z) = D_z^{-\rho} f(z),$$

where $D_z^{-\rho} f(z)$ is the fractional integral of order ρ of $f(z)$ which was introduced by Owa ([4], [5]).

In order to prove our results for the fractional integral operator, we have to recall here the following lemma due to Srivastava, Saigo and Owa [9].

Lemma 3. *If $\rho > 0$ and $n > \delta - \eta - 1$, then*

$$I_{0,z}^{\rho,\delta,\eta} z^n = \frac{\Gamma(n+1)\Gamma(n-\delta+\eta+1)}{\Gamma(n-\delta+1)\Gamma(n+\rho+\eta+1)} z^{n-\delta}. \tag{4.4}$$

With the aid of Lemma 3, we have

Theorem 10. *Let $\rho > 0$, $\delta < 2$, $\rho+\eta > -2$, $\delta-\eta < 2$, and $\delta(\rho+\eta) \leq 3\rho$. If $f(z) \in T$ is in the class $S^*(\alpha,\beta,\mu)$, then*

$$\left|I_{0,z}^{\rho,\delta,\eta} f(z)\right| \geq \frac{\Gamma(2-\delta+\eta)|z|^{1-\delta}}{\Gamma(2-\delta)\Gamma(2+\rho+\eta)} \left\{1 - \frac{2(1+\mu)\beta(1-\alpha)(2-\delta+\eta)}{D(2,\alpha,\beta,\mu)(2-\delta)(2+\rho+\eta)}|z|\right\} \tag{4.5}$$

and

$$\left|I_{0,z}^{\rho,\delta,\eta} f(z)\right| \leq \frac{\Gamma(2-\delta+\eta)|z|^{1-\delta}}{\Gamma(2-\delta)\Gamma(2+\rho+\eta)} \left\{1 + \frac{2(1+\mu)\beta(1-\alpha)(2-\delta+\eta)}{D(2,\alpha,\beta,\mu)(2-\delta)(2+\rho+\eta)}|z|\right\} \tag{4.6}$$

for $z \in U_0$, where

$$U_0 = \begin{cases} U & (\delta \leq 1) \\ U - \{0\} & (\delta > 1). \end{cases}$$

The equalities in (4.5) and (4.6) are attained by the function

$$f(z) = z - \frac{(1+\mu)\beta(1-\alpha)}{D(2,\alpha,\beta,\mu)} z^2. \tag{4.7}$$

Proof. By using Lemma 3, we have

$$\begin{aligned} I_{0,z}^{\rho,\delta,\eta} f(z) =& \frac{\Gamma(2-\delta+\eta)}{\Gamma(2-\delta)\Gamma(2+\rho+\eta)} z^{1-\delta} \\ & - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(n-\delta+\eta+1)}{\Gamma(n-\delta+1)\Gamma(n+\rho+\eta+1)} a_n z^{n-\delta}. \end{aligned} \tag{4.8}$$

Letting

$$\begin{aligned} H(z) =& \frac{\Gamma(2-\delta)\Gamma(2+\rho+\eta)}{\Gamma(2-\delta+\eta)} z^{\delta} I_{0,z}^{\rho,\delta,\eta} f(z) \\ =& z - \sum_{n=2}^{\infty} h(n) a_n z^n, \end{aligned} \tag{4.9}$$

where

$$h(n) = \frac{(2-\delta+\eta)_{n-1}(1)_n}{(2-\delta)_{n-1}(2+\rho+\eta)_{n-1}} \quad (n \geq 2), \tag{4.10}$$

we can see that $h(n)$ is non-increasing for integers $n \geq 2$, and we have

$$0 < h(n) \leq h(2) = \frac{2(2-\delta+\eta)}{(2-\delta)(2+\rho+\eta)}. \tag{4.11}$$

Since $f(z) \in S^*(\alpha,\beta,\mu)$, Lemma 1 implies that

$$D(2,\alpha,\beta,\mu)\sum_{n=2}^{\infty} a_n \leq \sum_{n=2}^{\infty} D(n,\alpha,\beta,\mu)a_n \leq (1+\mu)\beta(1-\alpha),$$

so that

$$\sum_{n=2}^{\infty} a_n \leq \frac{(1+\mu)\beta(1-\alpha)}{D(2,\alpha,\beta,\mu)}. \tag{4.12}$$

Therefore, by using (4.11) and (4.12), we have

$$\begin{aligned} |H(z)| &\geq |z| - h(2)|z|^2 \sum_{n=2}^{\infty} a_n \\ &\geq |z| - \frac{2(1+\mu)\beta(1-\alpha)(2-\delta+\eta)}{D(2,\alpha,\beta,\mu)(2-\delta)(2+\rho+\eta)}|z|^2 \end{aligned} \tag{4.13}$$

and

$$\begin{aligned} |H(z)| &\leq |z| + h(2)|z|^2 \sum_{n=2}^{\infty} a_n \\ &\leq |z| + \frac{2(1+\mu)\beta(1-\alpha)(2-\delta+\eta)}{D(2,\alpha,\beta,\mu)(2-\delta)(2+\rho+\eta)}|z|^2. \end{aligned} \tag{4.14}$$

This completes the proof of Theorem 10.

Similary, by applying Lemma 2 (instead of Lemma 1) to the function $f(z)$ belonging to the class $C^*(\alpha,\beta,\mu)$, we can derive

Theorem 11. *Let $\rho > 0$, $\delta < 2$, $\rho+\eta > -2$, $\delta-\eta < 2$, and $\delta(\rho+\eta) \leq 3\rho$. If $f(z) \in T$ is in the class $C^*(\alpha,\beta,\mu)$, then*

$$\begin{aligned} &\left|I_{0,z}^{\rho,\delta,\eta} f(z)\right| \\ &\geq \frac{\Gamma(2-\delta+\eta)|z|^{1-\delta}}{\Gamma(2-\delta)\Gamma(2+\rho+\eta)}\left\{1 - \frac{(1+\mu)\beta(1-\alpha)(2-\delta+\eta)}{D(2,\alpha,\beta,\mu)(2-\delta)(2+\rho+\eta)}|z|\right\} \end{aligned} \tag{4.15}$$

and

$$\begin{aligned} &\left|I_{0,z}^{\rho,\delta,\eta} f(z)\right| \\ &\leq \frac{\Gamma(2-\delta+\eta)|z|^{1-\delta}}{\Gamma(2-\delta)\Gamma(2+\rho+\eta)}\left\{1 + \frac{(1+\mu)\beta(1-\alpha)(2-\delta+\eta)}{D(2,\alpha,\beta,\mu)(2-\delta)(2+\rho+\eta)}|z|\right\} \end{aligned} \tag{4.16}$$

for $z \in U_0$*, where* U_0 *is defined in Theorem* 10*. The equalities in* (4.15) *and* (4.16) *are attained by the function*

$$f(z) = z - \frac{(1+\mu)\beta(1-\alpha)}{2D(2,\alpha,\beta,\mu)} z^2. \tag{4.17}$$

Remarks.

(1) Taking $\rho = -\delta = k$ in Theorems 10 and 11, we get the results of Theorems 3 and 4 obtained by Aouf [1], respectively.

(2) Putting $\mu = 1$ in Theorems 10 and 11, we get the corresponding results for the classes $S^*(\alpha,\beta)$ and $C^*(\alpha,\beta)$ studied by Gupta and Jain [2].

References

[1] M. K. Aouf, On subclasses of univalent functions with negative coefficients. III, Bull. Soc. Roy. Sci. Liége, 56:5-6 (1987), 465-473.

[2] V. P. Gupta and P. K. Jain, Certain classes of univalent functions with negative coefficients, Bull. Austral. Math. Soc., 14 (1976), 409-416.

[3] V. Kumar and S. L. Shukla, On a libera integral operator, Kyungpook Math. J., 24:1 (1984), 39-43.

[4] S. Owa, On the distortion theorems. I, Kyungpook Math., 18 (1978), 53-59.

[5] S. Owa, Some applications of the fractional calculus, Research Notes in Math. 187, Pitman, Boston, London and Melbourne, (1985), 164-175.

[6] S. Owa, On the classes of univalent functions with negative coefficients, Math. Japon., 27:4 (1982), 409-416.

[7] S. Owa and M. K. Aouf, On subclasses of univalent functinos with negative coefficients, Pusan Kyongnam Math. J., 4 (1988), 57-73.

[8] H. Silverman, Univalent functions with negative coefficients, Proc. Amer. Math. Soc., 51 (1975), 109-116.

[9] H. M. Srivastava, M. Saigo and S. Owa, A class of distortion theorems involving certain operators of fractional calculus, J. Math. Anal. Appl., 131 (1988), 412-420.

Department of Mathematics, Faculty of Science, University of Mansoura, Mansoura, Egypt.