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On certain classes of meromorphically multivalent functions associated with the generalized hypergeometric function

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ABSTRACT

Making use of a linear operator, which is defined here by means of the Hadamard product (or convolution), involving the generalized hypergeometric function, we introduce two novel subclasses $\Omega_{p,q,s}(\alpha_1; A, B, \lambda)$ and $\Omega^{+}_{p,q,s}(\alpha_1; A, B, \lambda)$ of meromorphically multivalent functions of order λ $(0 \le \lambda < p)$ in the punctured disc U^*. In this paper we investigate the various important properties and characteristics of these subclasses of meromorphically multivalent functions. We extend the familiar concept of neighborhoods of analytic functions to these subclasses of meromorphically multivalent functions. We also derive many interesting results for the Hadamard products of functions belonging to the class $\Omega^{+}_{p,q,s}(\alpha_1; A, B, \lambda)$.

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1. Introduction

Let $\sum_p$ denote the class of functions of the form:

$$f(z) = z^{-p} + \sum_{k=1}^{\infty} a_k z^{k-p} \quad (p \in N = \{1, 2, \ldots\}), \tag{1.1}$$

which are analytic and p-valent in the punctured disc

$$U^* = \{z : z \in C \text{ and } 0 < |z| < 1\} = U \setminus \{0\}.$$

For functions $f(z) \in \sum_p$ given by (1.1), and $g(z) \in \sum_p$ given by

$$g(z) = z^{-p} + \sum_{k=1}^{\infty} b_k z^{k-p} \quad (p \in N), \tag{1.2}$$

we define the Hadamard product (or convolution) of $f(z)$ and $g(z)$ by

$$(f * g)(z) = z^{-p} + \sum_{k=1}^{\infty} a_k b_k z^{k-p}. \tag{1.3}$$

For complex parameters

$$\alpha_1, \ldots, \alpha_q \quad \text{and} \quad \beta_1, \ldots, \beta_s \quad (\beta_j \notin Z_0^- = \{0, -1, -2, \ldots\}; j = 1, 2, \ldots, s),$$

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we now define the generalized hypergeometric function ${}_qF_s(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z)$ by

$$ {}_qF_s(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z)=\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\cdots(\alpha_q)_k}{(\beta_1)_k\cdots(\beta_s)_k}.\frac{z^k}{k!}\quad (q\le s+1;\, q,s\in N_0=N\cup\{0\};\, z\in U), \tag{1.4}$$

where $(\theta)_\nu$ is the Pochhammer symbol defined, in terms of the Gamma function Γ, by

$$(\theta)_\nu=\frac{\Gamma(\theta+\nu)}{\Gamma(\theta)}=\begin{cases}1, & (\nu=0;\,\theta\in C\setminus\{0\}),\\ \theta(\theta+1)\cdots(\theta+\nu-1), & (\nu\in N;\,\theta\in C).\end{cases} \tag{1.5}$$

Corresponding to the function $h_p(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z)$, defined by

$$h_p(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z)=z^{-p}\,{}_qF_s(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z),$$

Liu and Srivastava [1] (see, for details, [2] and [3]) introduced a linear operator:

$$H_p(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s):\Sigma_p\to\Sigma_p,$$

which is defined by the following Hadamard product (or convolution):

$$H_p(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s)f(z)=h_p(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z)*f(z). \tag{1.6}$$

We observe that, for a function $f(z)$ of the form (1.1), we have

$$H_p(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s)f(z)=z^{-p}+\sum_{k=1}^{\infty}\frac{(\alpha_1)_k\ldots(\alpha_q)_k}{(\beta_1)_k\ldots(\beta_s)_k}.\frac{a_k}{k!}z^{k-p}. \tag{1.7}$$

If, for convenience, we write

$$\begin{aligned}H_{p,q,s}(\alpha_1)&=H_p(\alpha_1,\ldots\alpha_q;\beta_1,\ldots,\beta_s),\\ H_{p,q,s}(\alpha_1+1)&=H_p(\alpha_1+1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s),\end{aligned} \tag{1.8}$$

where $\alpha_2,\ldots,\alpha_q,\beta_1,\ldots,\beta_s\in C$ remain fixed, then one can easily verify from the definition (1.6) that

$$z(H_{p,q,s}(\alpha_1)f(z))'=\alpha_1H_{p,q,s}(\alpha_1+1)f(z)-(\alpha_1+p)H_{p,q,s}(\alpha_1)f(z). \tag{1.9}$$

Some interesting subclasses of analytic functions, associated with the generalized hypergeometric function, were considered recently by (for example) Gangadharan et al. [4] and Liu [5].

Let $f(z)$ and $g(z)$ be analytic in U. Then we say that the function $g(z)$ is subordinate to $f(z)$ if there exists an analytic function $w(z)$ in U such that $w(0)=0$,

$$|w(z)|<1(z\in U)\quad\text{and}\quad g(z)=f(w(z)).$$

For this subordination, the symbol $g(z)\prec f(z)$ is used. In case $f(z)$ is univalent in U, the subordination $g(z)\prec f(z)$ is equivalent to

$$g(0)=f(0)\quad\text{and}\quad g(U)\subset f(U).$$

In this paper we generalize the classes $\Omega_{p,q,s}(\alpha_1;A,B)$ and $\Omega^+_{p,q,s}(\alpha_1;A,B)$ studied by Liu and Srivastava [1] as follows:

Making use of the operator $H_{p,q,s}(\alpha_1)$, we say that a function $f(z)\in\sum_p$ is in the class $\Omega_{p,q,s}(\alpha_1;A,B,\lambda)$ if it satisfies the following subordination condition:

$$\frac{1}{p-\lambda}\left(\frac{(H_{p,q,s}(\alpha_1+1)f(z))'}{(H_{p,q,s}(\alpha_1)f(z))'}-1\right)\prec-\frac{(A-B)z}{\alpha_1(1+Bz)},$$
$$(z\in U;\,-1\le B<A\le 1;\,\alpha_1\in C\setminus\{0\};\,p,q,s\in N;\,0\le\lambda<p), \tag{1.10}$$

or, equivalently, by using (1.9), if

$$\left|\frac{1+\frac{z(H_{p,q,s}(\alpha_1)f(z))''}{(H_{p,q,s}(\alpha_1)f(z))'}+p}{B\left(1+\frac{z(H_{p,q,s}(\alpha_1)f(z))''}{(H_{p,q,s}(\alpha_1)f(z))'}\right)+[pB+(A-B)(p-\lambda)]}\right|<1. \tag{1.11}$$

Furthermore, we introduce a second class $\Omega^+_{p,q,s}(\alpha_1;A,B,\lambda)$ defined as follows:

We say that a function $f(z)\in\Omega^+_{p,q,s}(\alpha_1;A,B,\lambda)$ wherever $f(z)$ is of the form:

$$f(z)=z^{-p}+\sum_{k=p}^{\infty}|a_k|z^k\quad(p\in N). \tag{1.12}$$

We note that: $\Omega_{p,q,s}(\alpha_1; A, B, 0) = \Omega_{p,q,s}(\alpha_1; A, B)$ and $\Omega^+_{p,q,s}(\alpha_1; A, B, 0) = \Omega^+_{p,q,s}(\alpha_1; A, B)$ (Liu and Srivastava [1]). Also we observe that:

$$\Omega^+_{p,q,s}(\alpha_1; \beta, -\beta, \lambda) = \Omega^+_{p,q,s}(\alpha_1; \lambda, \beta)$$
$$= \left\{ f(z) \in \Sigma_p \text{ and } \left| \frac{1 + \frac{z(H_{p,q,s}(\alpha_1)f(z))''}{(H_{p,q,s}(\alpha_1)f(z))'} + p}{1 + \frac{z(H_{p,q,s}(\alpha_1)f(z))''}{(H_{p,q,s}(\alpha_1)f(z))'} - p + 2\lambda} \right| < \beta \right.$$
$$\left. (z \in U; 0 \le \lambda < p; 0 < \beta \le 1; p \in N) \right\}. \tag{1.13}$$

Meromorphic multivalent functions have been extensively studied by (for example) Mogra [6,7], Uralegaddi and Ganigi [8], Uralegaddi and Somanatha [9], Aouf [10,11], Aouf and Hossen [12], Srivastava et al. [13], Owa et al. [14], Joshi and Aouf [15], Joshi and Srivastava [16], Aouf et al. [17], Raina and Srivastava [18], Yang [19,20], Kulkarni et al. [21], Liu [22] and Liu and Srivastava [23,24].

In this paper we investigate the various important properties and characteristics of the classes $\Omega_{p,q,s}(\alpha_1; A, B, \lambda)$ and $\Omega^+_{p,q,s}(\alpha_1; A, B, \lambda)$. Following the recent investigations by Altintas et al. [25, p. 1668], we extend the concept of neighborhoods of analytic functions, which was considered earlier by (for example) Goodman [26] and Ruscheweyh [27], to meromorphically multivalent functions, belonging to the classes $\Omega_{p,q,s}(\alpha_1; A, B, \lambda)$ and $\Omega^+_{p,q,s}(\alpha_1; A, B, \lambda)$. We also derive many interesting results for the Hadamard products of functions belonging to the p-valently meromorphic function class $\Omega^+_{p,q,s}(\alpha_1, A, B, \lambda)$.

The main results of the first class $\Omega_{p,q,s}(\alpha_1; A, B, \lambda)$ are mentioned in Theorems 1 and 2, respectively. Moreover, for the second class $\Omega^+_{p,q,s}(\alpha_1; A, B, \lambda)$, the main results will be included in Theorems 3 and 6–11, respectively.

2. Inclusion properties of the class $\Omega_{p,q,s}(\alpha_1; A, B, \lambda)$

We begin by recalling the following result (popularly known as Jack's lemma [28]), which we shall apply in proving our first inclusion theorem (Theorem 1).

Lemma 1 (*See [28]*). *Let the (nonconstant) function $w(z)$ be analytic in U with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point $z_0 \in U$, then*

$$z_0 w'(z_0) = \gamma w(z_0), \tag{2.1}$$

where γ is a real number and $\gamma \ge 1$.

Theorem 1. *Let $\alpha_1 \in R \setminus \{0\}$. If*

$$\alpha_1 \ge \frac{(A-B)(p-\lambda)}{1+B} \quad (-1 < B < A \le 1; 0 \le \lambda < p; p \in N), \tag{2.2}$$

then

$$\Omega_{p,q,s}(\alpha_1 + 1; A, B, \lambda) \subset \Omega_{p,q,s}(\alpha_1; A, B, \lambda).$$

Proof. Let $f(z) \in \Omega_{p,q,s}(\alpha_1 + 1; A, B, \lambda)$ and suppose that

$$\frac{(H_{p,q,s}(\alpha_1+1)f(z))'}{(H_{p,q,s}(\alpha_1)f(z))'} = 1 - \frac{(A-B)(p-\lambda)w(z)}{\alpha_1(1+Bw(z))}, \tag{2.3}$$

where the function $w(z)$ is either analytic or meromorphic in U, with $w(0) = 0$. By differentiating (2.3) with respect to z logarithmically and using (1.9), we have

$$(\alpha_1+1)\left(\frac{(H_{p,q,s}(\alpha_1+2)f(z))'}{(H_{p,q,s}(\alpha_1+1)f(z))'} - 1\right)$$
$$= \frac{(B-A)(p-\lambda)w(z)}{1+Bw(z)} + \frac{(B-A)(p-\lambda)zw'(z)}{(1+Bw(z))[\alpha_1 + (B\alpha_1 - (A-B)(p-\lambda))w(z)]}. \tag{2.4}$$

Now, if we suppose that

$$\max_{|z|\le|z_0|} |w(z)| = |w(z_0)| = 1 \quad (z_0 \in U), \tag{2.5}$$

and apply Jack's lemma, we find that

$$z_0 w'(z_0) = \gamma w(z_0) \quad (\gamma \ge 1). \tag{2.6}$$

Writing

$$w(z_0) = e^{i\theta} \quad (0 \leq \theta \leq 2\pi)$$

and putting $z = z_0$ in (2.4), we get after some computations that

$$\begin{aligned}
&\left|\frac{(\alpha_1+1)[(H_{p,q,s}(\alpha_1+2)f(z_0))' - (H_{p,q,s}(\alpha_1+1)f(z_0))']}{B(\alpha_1+1)[(H_{p,q,s}(\alpha_1+2)f(z_0))' - (H_{p,q,s}(\alpha_1+1)f(z_0))'] + D}\right|^2 - 1 \\
&= \left|\frac{(\alpha_1+\gamma) + [B\alpha_1 - (A-B)(p-\lambda)]e^{i\theta}}{\alpha_1 + [B(\alpha_1-\gamma) - (A-B)(p-\lambda)]e^{i\theta}}\right|^2 - 1 \\
&= \frac{\gamma^2(1-B^2) + 2\gamma[\alpha_1(1+B^2) - B(A-B)(p-\lambda)] + 2\gamma[2\alpha_1 B - (A-B)(p-\lambda)]\cos\theta}{\left|\alpha_1 + [B(\alpha_1-\gamma) - (A-B)(p-\lambda)]e^{i\theta}\right|^2},
\end{aligned} \tag{2.7}$$

where

$$D = (A-B)(p-\lambda)(H_{p,q,s}(\alpha_1+1)f(z_0))'.$$

Set

$$\begin{aligned}
&g(\theta) = \gamma^2(1-B^2) + 2\gamma[\alpha_1(1+B^2) - B(A-B)(p-\lambda)] + 2\gamma[2B\alpha_1 - (A-B)(p-\lambda)]\cos\theta \quad (0 \leq \theta \leq 2\pi) \\
&\quad (-1 < B < A \leq 1;\ 0 \leq \lambda < p;\ p \in N;\ \alpha_1 \in R \setminus \{0\};\ \gamma \geq 1;\ 0 \leq \theta \leq 2\pi).
\end{aligned} \tag{2.8}$$

Then, by hypothesis, we have

$$g(0) = \gamma^2(1-B^2) + 2\gamma(1+B)[\alpha_1(1+B) - (A-B)(p-\lambda)] \geq 0$$

and

$$g(\pi) = \gamma^2(1-B^2) + 2\gamma(1-B)[\alpha_1(1-B) + (A-B)(p-\lambda)] \geq 0$$

which, together, show that

$$g(\theta) \geq 0 \quad (0 \leq \theta \leq 2\pi). \tag{2.9}$$

In view of (2.9), (2.7) would obviously contradict our hypothesis that $f(z) \in \Omega_{p,q,s}(\alpha_1 + 1; A, B, \lambda)$. Hence, we must have

$$|w(z)| < 1 \quad (z \in U), \tag{2.10}$$

and we conclude from (2.3) that

$$f(z) \in \Omega_{p,q,s}(\alpha_1; A, B, \lambda).$$

The proof of Theorem 1 is thus completed. □

Next we prove an inclusion property associated with a certain integral transform introduced below.

Theorem 2. *Let μ be a complex number such that*

$$\mathrm{Re}(\mu) > \frac{(A-B)(p-\lambda)}{1+B} \quad (-1 < B < A \leq 1;\ 0 \leq \lambda < p;\ p \in N).$$

If $f(z) \in \Omega_{p,q,s}(\alpha_1; A, B, \lambda)$, then the function $F(z)$ defined by

$$F(z) = \frac{\mu}{z^{\mu+p}} \int_0^z t^{\mu+p-1} f(t)\,dt \tag{2.11}$$

also belongs to the class $\Omega_{p,q,s}(\alpha_1; A, B, \lambda)$.

Proof. From (2.11), we readily have

$$z(H_{p,q,s}(\alpha_1)F(z))' = \mu H_{p,q,s}(\alpha_1)f(z) - (\mu+p)H_{p,q,s}(\alpha_1)F(z). \tag{2.12}$$

Suppose that $f(z) \in \Omega_{p,q,s}(\alpha_1; A, B, \lambda)$ and put

$$\frac{(H_{p,q,s}(\alpha_1+1)F(z))'}{(H_{p,q,s}(\alpha_1)F(z))'} = 1 - \frac{[(A-B)(p-\lambda)]w(z)}{\alpha_1(1+Bw(z))}, \tag{2.13}$$

where the function $w(z)$ is either analytic or meromorphic in U, with $w(0) = 0$. Then, by using (2.12), (2.13) and the identity (1.9), we find after some calculations that

$$\alpha_1\left(\frac{(H_{p,q,s}(\alpha_1+1)f(z))'}{(H_{p,q,s}(\alpha_1)f(z))'}-1\right)=\frac{(B-A)(p-\lambda)w(z)}{1+Bw(z)}+\frac{(B-A)(p-\lambda)zw\prime(z)}{(1+Bw(z))[\mu+(\mu B-(A-B)(p-\lambda))w(z)]}.$$

The remaining part of the proof of Theorem 2 is similar to that of Theorem 1 and so is omitted. □

3. Properties of the class $\Omega^+_{p,q,s}(\alpha_1; A, B, \lambda)$

In this section we assume further that

$$\alpha_j > 0 \quad (j = 1, \dots, q), \qquad \beta_j > 0 \quad (j = 1, \dots, s), \quad 0 \le B < 1, \quad 0 \le \lambda < p, \quad \text{and} \quad p \in N.$$

We first determine a necessary and sufficient condition for a function $f(z) \in \Sigma_p$ of the form (1.12) to be in the class $\Omega^+_{p,q,s}(\alpha_1; A, B, \lambda)$ of meromorphically p-valent functions with positive coefficients.

Theorem 3. *Let $f(z) \in \Sigma_p$ be given by (1.12). Then $f(z) \in \Omega^+_{p,q,s}(\alpha_1; A, B, \lambda)$ if and only if*

$$\sum_{k=p}^{\infty} k\Gamma_{k+p}(\alpha_1)[(k+p)(1+B)+(A-B)(p-\lambda)]|a_k| \le p(A-B)(p-\lambda), \tag{3.1}$$

where, for convenience,

$$\Gamma_m(\alpha_1)=\frac{(\alpha_1)_m\dots(\alpha_q)_m}{(\beta_1)_m\dots(\beta_s)_m m!} \quad (m\in N). \tag{3.2}$$

Proof. Let $f(z) \in \Omega^+_{p,q,s}(\alpha_1; A, B, \lambda)$ be given by (1.12). Then from (1.11) and (1.12), we have

$$\begin{aligned}&\left|\frac{\alpha_1[(H_{p,q,s}(\alpha_1+1)f(z))'-(H_{p,q,s}(\alpha_1)f(z))']}{\alpha_1 B\,[(H_{p,q,s}(\alpha_1+1)f(z))'-(H_{p,q,s}(\alpha_1)f(z))']+(A-B)(p-\lambda)(H_{p,q,s}(\alpha_1)f(z))'}\right|\\ &\quad=\left|\frac{\sum_{k=p}^{\infty}k(k+p)\Gamma_{k+p}(\alpha_1)|a_k|z^{k+p}}{p(A-B)(p-\lambda)-\sum_{k=p}^{\infty}[B(k+p)+(A-B)(p-\lambda)]k\Gamma_{k+p}(\alpha_1)|a_k|z^{k+p}}\right|\\ &\quad< 1 \quad (z\in U).\end{aligned}$$

Since $|\mathrm{Re}(z)| \le |z| (z \in C)$, we have

$$\mathrm{Re}\left\{\frac{\sum_{k=p}^{\infty}k(k+p)\Gamma_{k+p}(\alpha_1)|a_k|z^{k+p}}{p(A-B)(p-\lambda)-\sum_{k=p}^{\infty}[B(k+p)+(A-B)(p-\lambda)]k\Gamma_{k+p}(\alpha_1)|a_k|z^{k+p}}\right\}<1 \quad (z\in U). \tag{3.3}$$

We consider real values of z and take $z = r$ with $0 \le r < 1$. Then, for $r = 0$, the denominator of (3.3) is positive and so is positive for all r $(0 < r < 1)$. Letting $z = r \to 1^-$, (3.3) yields

$$\sum_{k=p}^{\infty}k(k+p)\Gamma_{k+p}(\alpha_1)|a_k|\le p(A-B)(p-\lambda)-\sum_{k=p}^{\infty}[B(k+p)+(A-B)(p-\lambda)]k\Gamma_{k+p}(\alpha_1)|a_k|,$$

which leads us at once to (3.1).

In order to prove the converse, we assume that the inequality (3.1) holds true. Then we get

$$\begin{aligned}&\left|\frac{\alpha_1[(H_{p,q,s}(\alpha_1+1)f(z))'-(H_{p,q,s}(\alpha_1)f(z))']}{\alpha_1 B\left[(H_{p,q,s}(\alpha_1+1)f(z))'-(H_{p,q,s}(\alpha_1)f(z))'\right]+(A-B)(p-\lambda)(H_{p,q,s}(\alpha_1)f(z))'}\right|\\ &\quad\le\frac{\sum_{k=p}^{\infty}k(k+p)\Gamma_{k+p}(\alpha_1)|a_k|}{p(A-B)(p-\lambda)-\sum_{k=p}^{\infty}[B(k+p)+(A-B)(p-\lambda)]k\Gamma_{k+p}|a_k|}\\ &\quad<1 \quad (z\in U).\end{aligned}$$

Hence, by the maximum modulus theorem, we have $f(z) \in \Omega^+_{p,q,s}(\alpha_1; A, B, \lambda)$. This completes the proof of Theorem 3. □

Corollary 1. *Let $f(z) \in \Sigma_p$ be given by (1.12). If $f(z) \in \Omega^{+}_{p,q,s}(\alpha_1; A, B, \lambda)$, then*

$$|a_k| \leq \frac{p(A-B)(p-\lambda)}{k\Gamma_{k+p}(\alpha_1)\left[(k+p)(1+B)+(A-B)(p-\lambda)\right]} \quad (k \geq p;\, p \in N). \tag{3.4}$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z^{-p} + \frac{p(A-B)(p-\lambda)}{k\Gamma_{k+p}(\alpha_1)[(k+p)(1+B)+(A-B)(p-\lambda)]} z^k \quad (k \geq p;\, p \in N). \tag{3.5}$$

Putting $\lambda = 0$ in Theorem 3, we obtain

Corollary 2. *Let $f(z) \in \Sigma_p$ be given by (1.12). Then $f(z) \in \Omega^{+}_{p,q,s}(\alpha_1; A, B)$, if and only if*

$$\sum_{k=p}^{\infty} k\Gamma_{k+p}(\alpha_1)[(k+p)(1+B)+p(A-B)]\,|a_k| \leq p^2(A-B).$$

Remark 1. We note that the result obtained by Liu and Srivastava [1, Theorem 3] is not correct. The correct result is given by Corollary 2.

Next we prove the following growth and distortion properties for the class $\Omega^{+}_{p,q,s}(\alpha_1; A, B, \lambda)$.

Theorem 4. *Let the function $f(z)$ of the form (1.12) belong to the class $\Omega^{+}_{p,q,s}(\alpha_1; A, B, \lambda)$. If the sequence $[C_k]$ is nondecreasing, then*

$$r^{-p} - \frac{p(A-B)(p-\lambda)}{C_p} r^p \leq |f(z)| \leq r^{-p} + \frac{p(A-B)(p-\lambda)}{C_p} r^p \quad (0 < |z| = r < 1), \tag{3.6}$$

where

$$C_k = k\Gamma_{k+p}(\alpha_1)[(k+p)(1+B)+(A-B)(p-\lambda)] \quad (k \geq p;\, p \in N) \tag{3.7}$$

and $\Gamma_{k+p}(\alpha_1)$ is given by (3.2).

If the sequence $\{\frac{C_k}{k}\}$ is nondecreasing, then

$$pr^{-p-1} - \frac{p^2(A-B)(p-\lambda)}{C_p} r^{p-1} \leq \left|f'(z)\right| \leq pr^{-p-1} + \frac{p^2(A-B)(p-\lambda)}{C_p} r^{p-1} \quad (0 < |z| = r < 1). \tag{3.8}$$

Each of these results is sharp with the extremal function $f(z)$ given by

$$f(z) = z^{-p} + \frac{(A-B)(p-\lambda)}{\Gamma_{2p}(\alpha_1)[2p(1+B)+(A-B)(p-\lambda)]} z^p \quad (p \in N). \tag{3.9}$$

Proof. Let the function $f(z)$, given by (1.12), be in the class $\Omega^{+}_{p,q,s}(\alpha_1; A, B, \lambda)$. If the sequence $\{C_k\}$ is nondecreasing and positive, then, by Theorem 3, we have

$$\sum_{k=p}^{\infty} |a_k| \leq \frac{p(A-B)(p-\lambda)}{C_p}, \tag{3.10}$$

and if the sequence $\{\frac{C_k}{k}\}$ is nondecreasing and positive, Theorem 3 also yields

$$\sum_{k=p}^{\infty} k\,|a_k| \leq \frac{p^2(A-B)(p-\lambda)}{C_p}. \tag{3.11}$$

Making use of the conditions (3.10) and (3.11), in conjunction with the definition (1.12), we readily obtain the assertions (3.6) and (3.8) of Theorem 4.

Finally, it is easy to see that the bounds in (3.6) and (3.8) are attained for the function f(z) given by (3.9).

Next we determine the radii of meromorphically p-valent starlikeness of order ϕ $(0 \leq \phi < p)$ and meromorphically p-valent convexity of order ϕ $(0 \leq \phi < p)$ for functions in the class $\Omega^{+}_{p,q,s}(\alpha_1; A, B, \lambda)$. □

Theorem 5. *Let the function $f(z)$ defined by (1.12) be in the class $\Omega^{+}_{p,q,s}(\alpha_1; A, B, \lambda)$. Then* (i) *$f(z)$ is meromorphically p-valent starlike of order ϕ $(0 \le \phi < p)$ in the disc $|z| < r_1$, that is,*

$$\operatorname{Re}\left\{-\frac{zf'(z)}{f(z)}\right\} > \phi \quad (|z| < r_1; 0 \le \phi < p; p \in N), \tag{3.12}$$

where

$$r_1 = \inf_{k\ge p}\left\{\frac{k\Gamma_{k+p}(\alpha_1)(p-\phi)[(k+p)(1+B)+(A-B)(p-\lambda)]}{p(A-B)(p-\lambda)(k+\phi)}\right\}^{\frac{1}{k+p}}. \tag{3.13}$$

(ii) $f(z)$ is meromorphically p-valent convex of order ϕ $(0 \le \phi < p)$ in the disc $|z| < r_2$, that is,

$$\operatorname{Re}\left\{-\left(1+\frac{zf''(z)}{f'(z)}\right)\right\} > \phi \quad (|z| < r_2; 0 \le \phi < p; p \in N), \tag{3.14}$$

where

$$r_2 = \inf_{k\ge p}\left\{\frac{\Gamma_{k+p}(\alpha_1)(p-\phi)[(k+p)(1+B)+(A-B)(p-\lambda)]}{(A-B)(p-\lambda)(k+\phi)}\right\}^{\frac{1}{k+p}}, \tag{3.15}$$

and $\Gamma_{k+p}(\alpha_1)$ is given by (3.2). Each of these results is sharp for the function $f(z)$ given by (3.5).

Proof. (i) From the definition (1.12), we easily get

$$\left|\frac{\frac{zf'(z)}{f(z)}+p}{\frac{zf'(z)}{f(z)}-p+2\phi}\right| \le \frac{\sum_{k=p}^{\infty}(k+p)|a_k||z|^{k+p}}{2(p-\phi)-\sum_{k=p}^{\infty}(k-p+2\phi)|a_k||z|^{k+p}}. \tag{3.16}$$

Thus, we have the desired inequality:

$$\left|\frac{\frac{zf'(z)}{f(z)}+p}{\frac{zf'(z)}{f(z)}-p+2\phi}\right| \le 1 \quad (0 \le \phi < p; p \in N) \tag{3.17}$$

if

$$\sum_{k=p}^{\infty}\left(\frac{k+\phi}{p-\phi}\right)|a_k||z|^{k+p} \le 1. \tag{3.18}$$

Hence, by Theorem 3, (3.18) will be true if

$$\left(\frac{k+\phi}{p-\phi}\right)|z|^{k+p} \le \frac{k\Gamma_{k+p}(\alpha_1)[(k+p)(1+B)+(A-B)(p-\lambda)]}{p(A-B)(p-\lambda)} \quad (k \ge p; p \in N). \tag{3.19}$$

The last inequality (3.19) leads us immediately to the disc $|z| < r_1$, where r_1 is given by (3.13).

(ii) In order to prove the second assertion of Theorem 5, we find from the definition (1.12) that

$$\left|\frac{1+\frac{zf''(z)}{f'(z)}+p}{1+\frac{zf''(z)}{f'(z)}-p+2\phi}\right| \le \frac{\sum_{k=p}^{\infty}k(k+p)|a_k||z|^{k+p}}{2p(p-\phi)-\sum_{k=p}^{\infty}k(k-p+2\phi)|a_k||z|^{k+p}}. \tag{3.20}$$

Thus we have the desired inequality:

$$\left|\frac{1+\frac{zf''(z)}{f'(z)}+p}{1+\frac{zf''(z)}{f'(z)}-p+2\phi}\right| \le 1 \quad (0 \le \phi < p; p \in N), \tag{3.21}$$

if

$$\sum_{k=p}^{\infty}\frac{k(k+\phi)}{p(p-\phi)}|a_k||z|^{k+p} \le 1. \tag{3.22}$$

Hence, by Theorem 3, (3.22) will be true if

$$\frac{k(k+\phi)}{p(p-\phi)}|z|^{k+p} \leq \frac{k\Gamma_{k+p}(\alpha_1)[(k+p)(1+B)+(A-B)(p-\lambda)]}{p(A-B)(p-\lambda)} \quad (k \geq p;\, p \in N). \tag{3.23}$$

This last inequality (3.23) readily yields the disc $|z| < r_2$ with r_2 defined by (3.15), and the proof of Theorem 5 is completed by merely verifying that each assertion is sharp for the function $f(z)$ given by (3.5). □

4. Neighborhoods

In this section, we also assume that

$$\alpha_j > 0 \quad (j = 1, \ldots, q) \quad \text{and} \quad \beta_j > 0 \quad (j = 1, \ldots, s).$$

Following the earlier works (based upon the familiar concept of neighborhoods of analytic functions) by Goodman [26] and Ruscheweyh [27], and (more recently) by Altintas et al. [29,30,25], Liu [22], and Liu and Srivastava [23,24,1], we begin by introducing here the δ-neighborhood of a function $f(z) \in \Sigma_p$ of the form (1.1) by means of the definition given below:

$$N_\delta(f) = \Bigg\{ g : g \in \Sigma_p, g(z) = z^{-p} + \sum_{k=1}^{\infty} b_k z^{k-p} \text{ and } \sum_{k=1}^{\infty} \frac{(k+p)\Gamma_k(\alpha_1)[(A-B)(p-\lambda)+k(1+|B|)]}{p(A-B)(p-\lambda)} |a_k - b_k| \leq \delta \quad (-1 \leq B < A \leq 1;\, \delta > 0;\, 0 \leq \lambda < p;\, p \in N) \Bigg\}. \tag{4.1}$$

Making use of the definition (4.1), we now prove Theorem 6.

Theorem 6. *Let the function $f(z)$ defined by (1.1) be in the class $\Omega_{p,q,s}(\alpha_1; A, B, \lambda)$. If $f(z)$ satisfies the following condition:*

$$\frac{f(z) + \epsilon z^{-p}}{1+\epsilon} \in \Omega_{p,q,s}(\alpha_1; A, B, \lambda) \quad (\epsilon \in C, |\epsilon| < \delta, \delta > 0), \tag{4.2}$$

then

$$N_\delta(f) \subset \Omega_{p,q,s}(\alpha_1; A, B, \lambda). \tag{4.3}$$

Proof. It is easily seen from (1.11) that $g(z) \in \Omega_{p,q,s}(\alpha_1; A, B, \lambda)$ if and only if, for any complex σ with $|\sigma| = 1$,

$$\frac{1 + \frac{z(H_{p,q,s}(\alpha_1)g(z))''}{(H_{p,q,s}(\alpha_1)g(z))'} + p}{B\left(1 + \frac{z(H_{p,q,s}(\alpha_1)g(z))''}{(H_{p,q,s}(\alpha_1)g(z))'}\right) + [pB + (A-B)(p-\lambda)]} \neq \sigma \quad (z \in U;\ \sigma \in C;\, |\sigma| = 1), \tag{4.4}$$

which is equivalent to

$$\frac{(g * h)(z)}{z^{-p}} \neq 0 \quad (z \in U), \tag{4.5}$$

where, for convenience,

$$h(z) = z^{-p} + \sum_{k=1}^{\infty} c_k z^{k-p} = z^{-p} + \sum_{k=1}^{\infty} \frac{(k-p)\Gamma_k(\alpha_1)[(A-B)(p-\lambda)\sigma + k(B\sigma - 1)]}{p\sigma(B-A)(p-\lambda)} z^{k-p}. \tag{4.6}$$

From (4.6), we have

$$|c_k| = \left| \frac{(k-p)\Gamma_k(\alpha_1)[(A-B)(p-\lambda)\sigma + k(B\sigma - 1)]}{p\sigma(B-A)(p-\lambda)} \right| \leq \frac{(k+p)\Gamma_k(\alpha_1)[(A-B)(p-\lambda)+k(1+|B|)]}{p(A-B)(p-\lambda)} \quad (k, p \in N). \tag{4.7}$$

Now, if

$$f(z) = z^p + \sum_{k=1}^{\infty} a_k z^{k-p} \in \Sigma_p$$

satisfies the condition (4.2), then (4.5) yields

$$\left|\frac{(f*h)(z)}{z^{-p}}\right| \geq \delta \quad (z \in U; \delta > 0). \tag{4.8}$$

Let

$$g(z) = z^{-p} + \sum_{k=1}^{\infty} b_k z^{k-p} \in N_\delta(f), \tag{4.9}$$

so that

$$\begin{aligned} \left|\frac{[f(z)-g(z)]*h(z)}{z^{-p}}\right| &= \left|\sum_{k=1}^{\infty}(a_k-b_k)c_k z^k\right| \\ &\leq |z| \sum_{k=1}^{\infty} \frac{(k+p)\Gamma_k(\alpha_1)[(A-B)(p-\lambda)+k(1+|B|)]}{p(A-B)(p-\lambda)} b\,|a_k-b_k| \\ &< \delta \quad (z \in U; \delta > 0), \end{aligned} \tag{4.10}$$

which leads us to (4.5), and hence also (4.4) for any $\sigma \in C$ such that $|\sigma| = 1$. This implies that $g(z) \in \Omega_{p,q,s}(\alpha_1; A, B, \lambda)$, which evidently completes the proof of the assertion (4.3) of Theorem 6. □

We now define the δ-neighborhood of a function $f(z) \in \Sigma_p$ of the form (1.12) as follows:

$$\begin{aligned} N_\delta^+(f) = \Bigg\{ & g : g \in \Sigma_p, g(z) = z^{-p} + \sum_{k=p}^{\infty} |b_k| z^k \text{ and} \\ & \sum_{k=p}^{\infty} \frac{k\Gamma_{k+p}(\alpha_1)[(A-B)(p-\lambda)+(k+p)(1+B)]}{p(A-B)(p-\lambda)} ||a_k|-|b_k|| \leq \delta \\ & (0 \leq B < A \leq 1; \delta > 0; 0 \leq \lambda < p; p \in N) \Bigg\}. \end{aligned} \tag{4.11}$$

Theorem 7. *Let the function $f(z)$ defined by Eq. (1.12) be in the class $\Omega^+_{p,q,s}(\alpha_1+1; A, B, \lambda)(0 \leq B < A \leq 1; 0 \leq \lambda < p; p \in N)$. Then*

$$N_\delta^+(f) \subset \Omega^+_{p,q,s}(\alpha_1; A, B, \lambda) \quad \left(\delta = \frac{2p}{\alpha_1+2p}\right). \tag{4.12}$$

The result is sharp in the sense that δ cannot be increased.

Proof. Making use of the same method as in the proof of Theorem 6, we can show that [cf. Eq. (4.6)]

$$\begin{aligned} h(z) &= z^{-p} + \sum_{k=p}^{\infty} c_k z^k \\ &= z^{-p} + \sum_{k=p}^{\infty} \frac{k\Gamma_{k+p}(\alpha_1)[(A-B)(p-\lambda)\sigma+(k+p)(B\sigma-1)]}{p\sigma(B-A)(p-\lambda)} z^k. \end{aligned} \tag{4.13}$$

If $f(z) \in \Omega^+_{p,q,s}(\alpha_1+1; A, B, \lambda)$ is given by (1.12), we obtain

$$\begin{aligned} \left|\frac{(f*h)(z)}{z^{-p}}\right| &= \left|1 + \sum_{k=p}^{\infty} c_k |a_k| z^{k+p}\right| \\ &\geq 1 - \frac{\alpha_1}{\alpha_1+2p} \sum_{k=p}^{\infty} \frac{k\Gamma_{k+p}(\alpha_1+1)[(A-B)(p-\lambda)+(k+p)(1+B)]}{p(A-B)(p-\lambda)} |a_k| \\ &\geq 1 - \frac{\alpha_1}{\alpha_1+2p} = \frac{2p}{\alpha_1+2p} = \delta, \end{aligned}$$

by appealing to assertion (3.1) of Theorem 3. The remaining part of our proof of Theorem 7 is similar to that of Theorem 6, and we skip the details involved.

To show the sharpness of the assertion of Theorem 7, we consider the functions $f(z)$ and $g(z)$ given by

$$f(z)=z^{-p}+\frac{(A-B)(p-\lambda)}{\Gamma_{2p}(\alpha_1+1)[(A-B)(p-\lambda)+2p(1+B)]}z^{p}\in \Omega^{+}_{p,q,s}(\alpha_1+1;A,B,\lambda) \tag{4.14}$$

and

$$g(z)=z^{-p}+\left[\frac{(A-B)(p-\lambda)}{\Gamma_{2p}(\alpha_1+1)[(A-B)(p-\lambda)+2p(1+B)]}+\frac{(A-B)(p-\lambda)\delta'}{\Gamma_{2p}(\alpha_1)[(A-B)(p-\lambda)+2p(1+B)]}\right]z^{p}, \tag{4.15}$$

where

$$\delta'>\delta=\frac{2p}{\alpha_1+2p}.$$

Clearly, the function $g(z)$ belongs to $N^{+}_{\delta'}(f)$. On the other hand, we find from Theorem 3 that $g(z)$ is not in the class $\Omega^{+}_{p,q,s}(\alpha_1;A,B,\lambda)$. Thus the proof of Theorem 7 is completed. □

Finally, we prove the following theorem.

Theorem 8. *Let $f(z)\in\Sigma_p$ be given by (1.1) and define the partial sums $s_1(z)$ and $s_n(z)$ as follows:*

$$s_1(z)=z^{-p}\quad \textit{and}\quad s_n(z)=z^{-p}+\sum_{k=1}^{n-1}a_k z^{k-p}\quad (n\in N), \tag{4.16}$$

it being understood that an empty sum is (as usual) nil. Suppose also that

$$\sum_{k=1}^{\infty}d_k|a_k|\le 1\quad \left(d_k=\frac{(k+p)\Gamma_k(\alpha_1)[(A-B)(p-\lambda)+k(1+|B|)]}{p(A-B)(p-\lambda)}\right). \tag{4.17}$$

Then

(i) $f(z)\in\Omega_{p,q,s}(\alpha_1;A,B,\lambda)$,

(ii) If $\{\Gamma_k(\alpha_1)\}(k\in N)$ is nondecreasing and

$$\Gamma_1(\alpha_1)>\frac{p(A-B)(p-\lambda)}{(1+p)[(A-B)(p-\lambda)+(1+|B|)]}, \tag{4.18}$$

then

$$\mathrm{Re}\left\{\frac{f(z)}{s_n(z)}\right\}>1-\frac{1}{d_n}\quad (z\in U;n\in N), \tag{4.19}$$

and

$$\mathrm{Re}\left\{\frac{s_n(z)}{f(z)}\right\}>\frac{d_n}{1+d_n}\quad (z\in U;n\in N). \tag{4.20}$$

Each of the bounds in (4.19) and (4.20) is the best possible for each $n\in N$.

Proof. (i) It is not difficult to see that $z^{-p}\in\Omega_{p,q,s}(\alpha_1;A,B,\lambda)(p\in N)$. Thus, from Theorem 6 and the hypothesis (4.17) of Theorem 8, we have

$$N_1(z^{-p})\subset\Omega_{p,q,s}(\alpha_1;A,B,\lambda)\quad (0\le\lambda<p;p\in N), \tag{4.21}$$

which shows that $f(z)\in\Omega_{p,q,s}(\alpha_1;A,B,\lambda)$.

(ii) Under the hypothesis in Part (ii) of Theorem 8, we can see from (4.17) that

$$d_{k+1}>d_k>1\quad (k\in N). \tag{4.22}$$

Therefore, we have

$$\sum_{k=1}^{n-1}|a_k|+d_n\sum_{k=n}^{\infty}|a_k|\le\sum_{k=1}^{\infty}d_k|a_k|\le 1, \tag{4.23}$$

where we have used the hypothesis (4.17) again.

By setting

$$g_1(z)=d_n\left[\frac{f(z)}{s_n(z)}-\left(1-\frac{1}{d_n}\right)\right]=1+\frac{d_n\sum\limits_{k=n}^{\infty}a_kz^k}{1+\sum\limits_{k=1}^{n-1}a_kz^k}, \tag{4.24}$$

and applying (4.23), we find that

$$\left|\frac{g_1(z)-1}{g_1(z)+1}\right|\le\frac{d_n\sum\limits_{k=n}^{\infty}|a_k|}{2-2\sum\limits_{k=1}^{n-1}|a_k|-d_n\sum\limits_{k=n}^{\infty}|a_k|}\le 1\quad(z\in U), \tag{4.25}$$

which readily yields the assertion (4.19) of Theorem 8. If we take

$$f(z)=z^{-p}-\frac{z^{n-p}}{d_n}, \tag{4.26}$$

then

$$\frac{f(z)}{s_n(z)}=1-\frac{z^n}{d_n}\to 1-\frac{1}{d_n}\quad(z\to 1^-),$$

which shows that the bound in (4.19) is the best possible for each $n\in N$. Similarly, if we put

$$\begin{aligned} g_2(z) &= (1+d_n)\left(\frac{s_n(z)}{f(z)}-\frac{d_n}{1+d_n}\right)\\ &= 1-\frac{(1+d_n)\sum\limits_{k=n}^{\infty}a_kz^k}{1+\sum\limits_{k=1}^{\infty}a_kz^k} \end{aligned} \tag{4.27}$$

and make use of (4.23), we can deduce that

$$\left|\frac{g_2(z)-1}{g_2(z)+1}\right|\le\frac{(1+d_n)\sum\limits_{k=n}^{\infty}|a_k|}{2-2\sum\limits_{k=1}^{n-1}|a_k|+(1-d_n)\sum\limits_{k=n}^{\infty}|a_k|}\le 1\quad(z\in U), \tag{4.28}$$

which leads us immediately to the assertion (4.20) of Theorem 8.

The bound in (4.20) is sharp for each $n\in N$, with the extremal function $f(z)$ given by (4.26). The proof of Theorem 8 is thus completed. □

5. Convolution properties

For the functions

$$f_j(z)=z^{-p}+\sum_{k=p}^{\infty}|a_{k,j}|z^k\quad(j=1,2;\ p\in N), \tag{5.1}$$

we denote by $(f_1\circledast f_2)(z)$ the Hadamard product (or convolution) of the functions $f_1(z)$ and $f_2(z)$, that is,

$$(f_1\circledast f_2)(z)=z^{-p}+\sum_{k=p}^{\infty}|a_{k,1}||a_{k,2}|z^k. \tag{5.2}$$

Throughout this section, we assume further that the sequence $\{\Gamma_m(\alpha_1)\}(m\in N)$ is nondecreasing, where $\Gamma_m(\alpha_1)$ is given by (3.2),

$$C(p,\lambda,A,B,k)=(k+p)(1+B)+(A-B)(p-\lambda)\quad(k\ge p) \tag{5.3}$$

and

$$D(p,\lambda,A,B)=p(A-B)(p-\lambda). \tag{5.4}$$

Theorem 9. *Let the functions* $f_j(z)$ $(j = 1, 2)$ *defined by* (5.1) *be in the class* $\Omega^+_{p,q,s}(\alpha_1; A, B, \lambda)$. *Then* $(f_1 \circledast f_2)(z) \in \Omega^+_{p,q,s}(\alpha_1; A, B, \gamma)$, *where*

$$\gamma = p\left(1 - \frac{2(1+B)(A-B)(p-\lambda)^2}{\Gamma_{2p}(\alpha_1)[2p(1+B)+(A-B)(p-\lambda)]^2 - (A-B)^2(p-\lambda)^2}\right). \tag{5.5}$$

The result is sharp for the functions $f_j(z)$ $(j = 1, 2)$ given by

$$f_j(z) = z^{-p} + \frac{(A-B)(p-\lambda)}{\Gamma_{2p}(\alpha_1)[2p(1+B)+(A-B)(p-\lambda)]} z^p \quad (j = 1, 2; p \in N). \tag{5.6}$$

Proof. Employing the technique used earlier by Schild and Silverman [31], we need to find the largest γ such that

$$\sum_{k=p}^{\infty} \frac{k\Gamma_{k+p}(\alpha_1)C(p, \gamma, A, B, k)}{D(p, \gamma, A, B)} |a_{k,1}||a_{k,2}| \le 1 \tag{5.7}$$

for $f_j(z) \in \Omega^+_{p,q,s}(\alpha_1; A, B, \lambda)$ $(j = 1, 2)$. Since $f_j(z) \in \Omega^+_{p,q,s}(\alpha_1; A, B, \lambda)$ $(j = 1, 2)$, we readily see that

$$\sum_{k=p}^{\infty} \frac{k\Gamma_{k+p}(\alpha_1)C(p, \lambda, A, B, k)}{D(p, \lambda, A, B)} |a_{k,j}| \le 1 \quad (j = 1, 2). \tag{5.8}$$

Therefore, by the Cauchy–Schwarz inequality, we obtain

$$\sum_{k=p}^{\infty} \frac{k\Gamma_{k+p}(\alpha_1)C(p, \lambda, A, B, k)}{D(p, \lambda, A, B)} \sqrt{|a_{k,1}||a_{k,2}|} \le 1. \tag{5.9}$$

This implies that, we only need to show that

$$\frac{C(p, \gamma, A, B, k)}{(p-\gamma)} |a_{k,1}||a_{k,2}| \le \frac{C(p, \lambda, A, B, k)}{(p-\lambda)} \sqrt{|a_{k,1}||a_{k,2}|} \quad (k \ge p) \tag{5.10}$$

or, equivalently, that

$$\sqrt{|a_{k,1}||a_{k,2}|} \le \frac{(p-\gamma)C(p, \lambda, A, B, k)}{(p-\lambda)C(p, \gamma, A, B, k)} \quad (k \ge p). \tag{5.11}$$

Hence, by the inequality (5.9), it is sufficient to prove that

$$\frac{D(p, \lambda, A, B)}{k\Gamma_{k+p}(\alpha_1)C(p, \lambda, A, B, k)} \le \frac{(p-\gamma)C(p, \lambda, A, B, k)}{(p-\lambda)C(p, \gamma, A, B, k)} \quad (k \ge p). \tag{5.12}$$

It follows from (5.12) that

$$\gamma \le p - \frac{p(k+p)(1+B)(A-B)(p-\lambda)^2}{k\Gamma_{p+k}(\alpha_1)[C(p, \lambda, A, B, k)]^2 - p(A-B)^2(p-\lambda)^2} \quad (k \ge p). \tag{5.13}$$

Now, defining the function $\Phi(k)$ by

$$\Phi(k) = p - \frac{p(k+p)(1+B)(A-B)(p-\lambda)^2}{k\Gamma_{p+k}(\alpha_1)[C(p, \lambda, A, B, k)]^2 - p(A-B)^2(p-\lambda)^2} \quad (k \ge p), \tag{5.14}$$

we see that $\Phi(k)$ is an increasing function of k. Therefore, we conclude that

$$\gamma \le \Phi(p) = p\left(1 - \frac{2(1+B)(A-B)(p-\lambda)^2}{\Gamma_{2p}(\alpha_1)[2p(1+B)+(A-B)(p-\lambda)]^2 - (A-B)^2(p-\lambda)^2}\right), \tag{5.15}$$

which evidently completes the proof of Theorem 9. □

Putting $A = \beta$ and $B = -\beta$ $(0 < \beta \leq 1)$ in Theorem 9, we obtain the following consequence.

Corollary 3. *Let the functions $f_j(z)$ $(j = 1, 2)$ defined by (5.1) be in the class $\Omega^{+}_{p,q,s}(\alpha_1; \lambda, \beta)$. Then $(f_1 \circledast f_2)(z) \in \Omega^{+}_{p,q,s}(\alpha_1; \gamma, \beta)$, where*

$$\gamma = p\left(1 - \frac{\beta(1-\beta)(p-\lambda)^2}{\Gamma_{2p}(\alpha_1)(p-\lambda\beta)^2 - \beta^2(p-\lambda)^2}\right). \tag{5.16}$$

The result is sharp for the functions $f_j(z)$ $(j = 1, 2)$ given by

$$f_j(z) = z^{-p} + \frac{\beta(p-\lambda)}{\Gamma_{2p}(\alpha_1)(p-\lambda\beta)} z^p \quad (j = 1, 2; p \in N). \tag{5.17}$$

Using arguments similar to those in the proof of Theorem 9, we obtain the following result.

Theorem 10. *Let the function $f_1(z)$ defined by (5.1) be in the class $\Omega^{+}_{p,q,s}(\alpha_1; A, B, \lambda)$. Suppose also that the function $f_2(z)$ defined by (5.1) be in the class $\Omega^{+}_{p,q,s}(\alpha_1; A, B, \gamma)$. Then $(f_1 \circledast f_2)(z) \in \Omega^{+}_{p,q,s}(\alpha_1; A, B, \xi)$, where*

$$\xi = p\left(1 - \frac{2(1+B)(A-B)(p-\lambda)(p-\gamma)}{\Gamma_{2p}(\alpha_1)[2p(1+B)+(A-B)(p-\lambda)][2p(1+B)+(A-B)(p-\gamma)] - \Omega}\right)$$
$$(\Omega = (A-B)^2(p-\lambda)(p-\gamma)). \tag{5.18}$$

The result is sharp for the functions $f_j(z)$ $(j = 1, 2)$ given by

$$f_1(z) = z^{-p} + \frac{(A-B)(p-\lambda)}{\Gamma_{2p}(\alpha_1)[2p(1+B)+(A-B)(p-\lambda)]} z^p \quad (p \in N) \tag{5.19}$$

and

$$f_2(z) = z^{-p} + \frac{(A-B)(p-\gamma)}{\Gamma_{2p}(\alpha_1)[2p(1+B)+(A-B)(p-\gamma)]} z^p \quad (p \in N). \tag{5.20}$$

Putting $A = \beta$ and $B = -\beta$ $(0 < \beta \leq 1)$ in Theorem 10, we obtain Corollary 4.

Corollary 4. *Let the function $f_1(z)$ defined by (5.1) be in the class $\Omega^{+}_{p,q,s}(\alpha_1; \lambda, \beta)$. Suppose also that the function $f_2(z)$ defined by (5.1) be in the class $\Omega^{+}_{p,q,s}(\alpha_1; \gamma, \beta)$. Then $(f_1 \circledast f_2)(z) \in \Omega^{+}_{p,q,s}(\alpha_1; \eta, \beta)$, where*

$$\eta = p\left(1 - \frac{\beta(1-\beta)(p-\lambda)(p-\gamma)}{\Gamma_{2p}(\alpha_1)(p-\lambda\beta)(p-\gamma\beta) - \beta^2(p-\lambda)(p-\gamma)}\right). \tag{5.21}$$

The result is the best possible for the functions $f_j(z)$ $(j = 1, 2)$ given by

$$f_1(z) = z^{-p} + \frac{\beta(p-\lambda)}{\Gamma_{2p}(\alpha_1)(p-\lambda\beta)} z^p \quad (p \in N) \tag{5.22}$$

and

$$f_2(z) = z^{-p} + \frac{\beta(p-\gamma)}{\Gamma_{2p}(\alpha_1)(p-\gamma\beta)} z^p \quad (p \in N). \tag{5.23}$$

Theorem 11. *Let the functions $f_j(z)$ $(j = 1, 2)$ defined by (5.1) be in the class $\Omega^{+}_{p,q,s}(\alpha_1; A, B, \lambda)$. Then the function $h(z)$ defined by*

$$h(z) = z^{-p} + \sum_{k=p}^{\infty} (|a_{k,1}|^2 + |a_{k,2}|^2) z^k \tag{5.24}$$

belongs to the class $\Omega^{+}_{p,q,s}(\alpha_1; A, B, \zeta)$, where

$$\zeta = p\left(1 - \frac{4(1+B)(A-B)(p-\lambda)^2}{\Gamma_{2p}(\alpha_1)[2p(1+B)+(A-B)(p-\lambda)]^2 - 2(A-B)^2(p-\lambda)^2}\right). \tag{5.25}$$

This result is sharp for the functions $f_j(z)$ $(j = 1, 2)$ given already by (5.6).

Proof. Noting that

$$\sum_{k=p}^{\infty} \frac{[k\Gamma_{k+p}(\alpha_1)C(p,\lambda,A,B,k)]^2}{[D(p,\lambda,A,B)]^2}|a_{k,j}|^2 \le \left(\sum_{k=p}^{\infty} \frac{k\Gamma_{k+p}(\alpha_1)C(p,\lambda,A,B,k)}{D(p,\lambda,A,B)}|a_{k,j}|\right)^2 \le 1 \quad (j=1,2), \tag{5.26}$$

for $f_j(z) \in \Omega^{+}_{p,q,s}(\alpha_1; A, B, \lambda)$ $(j = 1, 2)$, we have

$$\sum_{k=p}^{\infty} \frac{[k\Gamma_{k+p}(\alpha_1)C(p,\lambda,A,B,k)]^2}{2[D(p,\lambda,A,B)]^2} (|a_{k,1}|^2 + |a_{k,2}|^2) \le 1. \tag{5.27}$$

Therefore, we have to find the largest ζ such that

$$\frac{C(p,\zeta,A,B,k)}{(p-\zeta)} \le \frac{k\Gamma_{k+p}(\alpha_1)[C(p,\lambda,A,B,k)]^2}{2p(A-B)(p-\lambda)^2} \quad (k \ge p), \tag{5.28}$$

that is,

$$\zeta \le p - \frac{2p(k+p)(1+B)(A-B)(p-\lambda)^2}{k\Gamma_{k+p}(\alpha_1)[C(p,\lambda,A,B,k)]^2 - 2p(A-B)^2(p-\lambda)^2} \quad (k \ge p). \tag{5.29}$$

Now, defining a function $\Psi(k)$ by

$$\Psi(k) = p - \frac{2p(k+p)(1+B)(A-B)(p-\lambda)^2}{k\Gamma_{k+p}(\alpha_1)[C(p,\lambda,A,B,k)]^2 - 2p(A-B)^2(p-\lambda)^2} \quad (k \ge p), \tag{5.30}$$

we observe that $\Psi(k)$ is an increasing function of k. We thus conclude that

$$\zeta \le \Psi(p) = p\left(1 - \frac{4(1+B)(A-B)(p-\lambda)^2}{\Gamma_{2p}(\alpha_1)[2p(1+B)+(A-B)(p-\lambda)]^2 - 2(A-B)^2(p-\lambda)^2}\right), \tag{5.31}$$

which completes the proof of Theorem 11. □

Putting $A = \beta$ and $B = -\beta$ $(0 < \beta \le 1)$ in Theorem 11, we obtain the following corollary.

Corollary 5. *Let the functions $f_j(z)$ $(j = 1, 2)$ defined by (5.1) be in the class $\Omega^{+}_{p,q,s}(\alpha_1; \lambda, \beta)$. Then the function $h(z)$ defined by (5.24) belongs to the class $\Omega^{+}_{p,q,s}(\alpha_1; \tau, \beta)$, where*

$$\tau = p\left(1 - \frac{2\beta(1-\beta)(p-\lambda)^2}{\Gamma_{2p}(\alpha_1)(p-\lambda\beta)^2 - 2\beta^2(p-\lambda)^2}\right). \tag{5.32}$$

The result is sharp for the functions $f_j(z)$ $(j = 1, 2)$ given already by (5.17).

Remark 2. We note that the results obtained by Liu and Srivastava [1, Theorems 4, 5 and 7], are not correct. The correct results are given by Theorems 4, 5 and 7, respectively, after putting $\lambda = 0$.

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