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# SOME APPLICATIONS OF FRACTIONAL CALCULUS OPERATORS TO A CERTAIN SUBCLASSES OF UNIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

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**Abstract**. The object of the present paper is to prove various distortion theorems for the fractional calculus and fractional integral operator of functions in the class $P^*(\alpha,\beta,\mu)$, consisting of analytic and univalent functions with negative coefficients.

## 1. Introduction

Let $S$ denote the class of functions of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n,\quad (a_n\geq 0),\tag{1.1}$$

which are analytic and univalent in the unit disc $U=\{z:|z|<1\}$. Also let $T$ be the subclass of $S$ consists of functions in the form

$$f(z)=z-\sum_{n=2}^{\infty}a_nz^n,\quad (a_n\geq 0).\tag{1.2}$$

A function $f(z)\in T$ is said to be in the class $P^*(\alpha,\beta,\mu)$ if and only if

$$\left|\frac{f'(z)-1}{\mu f'(z)+1-(1+\mu)\alpha}\right|<\beta,\quad (z\in U),\tag{1.3}$$

for some $\alpha(0\leq\alpha<1)$, $\beta(0<\beta\leq 1)$ and $\mu(0\leq\mu\leq 1)$. The class $P^*(\alpha,\beta,\mu)$ was studied by Owa and Aouf [4]. In particular the class $P^*(\alpha,\beta,1)=P^*(\alpha,\beta)$ was studied by Gupta and Jain [1].

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In order to show our results, we shall need the following lemma given by Owa and Aouf [4].

**Lemma 1.** *Let the function* $f(z)$ *be defined by* (1.2). *Then* $f(z) \in P^*(\alpha, \beta, \mu)$ *if and only if*

$$\sum_{n=2}^{\infty} n(1+\mu\beta)a_n \leq (1+\mu)\beta(1-\alpha). \tag{1.4}$$

*The result is sharp.*

## 2. Fractional Calculus and Fractional Integral Operator

We begin with the statements of the following definitions of fractional calculus (that is, fractional derivatives and fractional integrals) which were defined by Owa [2], [3] and were used recently by Srivastava and Owa [6], also definition of fractioal integral operator given bye Srivastava, Saigo and Owa [7] and Srivastara and Owa [5].

**Definition 1.** The fractional integral of order $k$ is defind, for a function $f(z)$, by

$$D_z^{-k} f(z) = \frac{1}{\Gamma(k)} \int_0^z \frac{f(\varsigma)}{(z-\varsigma)^{1-k}} d\varsigma, \quad (k > 0), \tag{2.1}$$

where $f(z)$ is an analytic function in a simply-connected region of the $z$-plane containing the origin, and the multiplicity of $(z-\varsigma)^{k-1}$ is removed by requiring $\log(z-\varsigma)$ to be real when $z - \varsigma > 0$.

**Definition 2.** The fractional derivative of order $k$ is defined, for a function $f(z)$, by

$$D_z^{k} f(z) = \frac{1}{\Gamma(1-k)} \frac{d}{dz} \int_0^z \frac{f(\varsigma)}{(z-\varsigma)^{k}} d\varsigma, \quad (0 \leq k < 1), \tag{2.2}$$

where $f(z)$ is constrained, and the multiplicity of $(z-\varsigma)^{-k}$ is removed, as in Definition 1.

**Definition 3.** Under the hypotheses of Definition 2, the fractional derivative of order $n+k$ is defined by

$$D_z^{n+k} f(z) = \frac{d^n}{dz^n} D_z^k f(z), \quad (0 \leq k < 1; n \in \{0, 1, \ldots\}). \tag{2.3}$$

**Definition 4.** For real numbers $\rho > 0$, $\delta$ and $\eta$, the fractional operator $I_{0,z}^{\rho,\delta,\eta}$ is defined by

$$I_{0,z}^{\rho,\delta,\eta} f(z) = \frac{z^{-\rho-\delta}}{\Gamma(\rho)} \int_0^z (z-\varsigma)^{\rho-1} F(\rho+\delta, -\eta; \rho; 1-\frac{\varsigma}{z}) f(\varsigma) d\varsigma, \tag{2.4}$$

where $f(z)$ is an analytic function in a simply connected region of the $z-$plane containing the origin with the order

$$f(z) = O(|z|^{\varepsilon}), z \to 0,$$

where

$$\varepsilon > \text{Max}(0, \delta-\eta) - 1,$$

$$F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n, \tag{2.5}$$

where $(\theta)_n$ is the Pochhammer symbol defined by

$$(\theta)_n = \frac{\Gamma(\theta+n)}{\Gamma(\theta)} = \begin{cases} 1 & (n=0), \\ \theta(\theta+1)\cdots(\theta+n-1) & (n \in N = \{1,2,\ldots\}), \end{cases} \tag{2.6}$$

and the multiplicity of $(z-\xi)^{p-1}$ is removed as in Definition 1.

**Remark.** For $\delta = -\rho$, we note that

$$I_{0,z}^{\rho,-\rho,\eta} f(z) = D_z^{-\rho} f(z).$$

In order to prove our result for the fractional integral operator, we have to recall here the following lemma due to Srivastava, Saigo and Owa [7].

**Lemma 2.** *If $\rho > 0$ and $n > \delta - \eta - 1$, then*

$$I_{0,z}^{\rho,\delta,\eta} z^n = \frac{\Gamma(n+1)\Gamma(n-\delta+\eta+1)}{\Gamma(n-\delta+1)\Gamma(n+\delta+\eta+1)} z^{n-\delta}. \tag{2.7}$$

With the aid of Lemma 2, we have

**Theorem 1.** *Let $\rho > 0$, $\delta < 2$, $\rho+\eta > -2$, $\delta-\eta < 2$, $\delta(\rho+\eta) \leq 3\rho$. If the function $f(z)$ defined by (1.2) is in the class $P^*(\alpha,\beta,\mu)$, then*

$$\left| I_{0,z}^{\rho,\delta,\eta} f(z) \right| \geq \frac{\Gamma(2-\delta+\eta)|z|^{1-\delta}}{\Gamma(2-\delta)\Gamma(2+\rho+\eta)} \left\{ 1 - \frac{(1+\mu)\beta(1-\alpha)(2-\delta+\eta)}{(1+\mu\beta)(2-\delta)(2+\rho+\eta)} |z| \right\} \tag{2.8}$$

*and*

$$\left| I_{0,z}^{\rho,\delta,\eta} f(z) \right| \leq \frac{\Gamma(2-\delta+\eta)|z|^{1-\delta}}{\Gamma(2-\delta)\Gamma(2+\rho+\eta)} \left\{ 1 + \frac{(1+\mu)\beta(1-\alpha)(2-\delta+\eta)}{(1+\mu\beta)(2-\delta)(2+\rho+\eta)} |z| \right\} \tag{2.9}$$

*for* $z \in U_0$, *where*

$$U_0 = \begin{cases} U & (\delta \leq 1), \\ U - \{0\} & (\delta > 1). \end{cases}$$

*The equalities in* (2.8) *and* (2.9) *are attained by the function* $f(z)$ *defined by*

$$f(z) = z - \frac{(1+\mu)\beta(1-\alpha)}{2(1+\mu\beta)} z^2. \tag{2.10}$$

**Proof.** by using Lemma 2, we have

$$I_{0,z}^{\rho,\delta,\eta} f(z) = \frac{\Gamma(2-\delta+\eta)}{\Gamma(2-\delta)\Gamma(2+\rho+\eta)} z^{1-\delta} - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(n-\delta+\eta+1)}{\Gamma(n-\delta+1)\Gamma(n+\rho+\eta+1)} a_n z^{n-\delta}. \tag{2.11}$$

Letting

$$H(z) = \frac{\Gamma(2-\delta)\Gamma(2+\rho+\eta)}{\Gamma(2-\delta+\eta)} z^{\delta} I_{0,z}^{\rho,\delta,\eta} f(z) = z - \sum_{n=2}^{\infty} h(n) a_n z^n, \tag{2.12}$$

where

$$h(n) = \frac{(2-\delta+\eta)_{n-1}(1)_n}{(2-\delta)_{n-1}(2+\rho+\eta)_{n-1}} (n \geq 2), \tag{2.13}$$

we can see that $h(n)$ is non-increasing for integers $n (n \geq 2)$, and we have

$$0 < h(n) \leq h(2) = \frac{(2(2-\delta+\eta))}{(2-\delta)(2+\rho+\eta)}. \tag{2.14}$$

In view of Lemma 1, we have

$$2(1+\mu\beta) \sum_{n=2}^{\infty} a_n \leq \sum_{n=2}^{\infty} n(1+\mu\beta) a_n \leq (1+\mu)\beta(1-\alpha),$$

which evidently yields

$$\sum_{n=2}^{\infty} a_n \leq \frac{(1+\mu)\beta(1-\alpha)}{2(1+\mu\beta)}. \tag{2.15}$$

Therefore, by using (2.14) and (2.15), we have

$$\begin{aligned} |H(z)| &\geq |z| - h(2)|z|^2 \sum_{n=2}^{\infty} a_n \\ &\geq |z| - \frac{(1+\mu)\beta(1-\alpha)(2-\delta+\eta)}{(1+\mu\beta)(2-\delta)(2+\rho+\eta)} |z|^2, \end{aligned} \tag{2.16}$$

and

$$|H(z)| \leq |z| + h(z)|z|^2 \sum_{n=2}^{\infty} a_n$$
$$\leq |z| + \frac{(1+\mu)\beta(1-\alpha)(2-\delta+\eta)}{(1+\mu\beta)(2-\delta)(2+\ell+\eta)}|z|^2. \tag{2.17}$$

This completes the proof of Theorem 1.

**Theorem 2.** *Let the function $f(z)$ defined by* (1.2) *be in the class $P^*(\alpha, \beta, \mu)$. Then we have*

$$\left|D_z^{-k} f(z)\right| \geq \frac{|z|^{1+k}}{\Gamma(2+k)} \left\{1 - \frac{(1+\mu)\beta(1-\alpha)}{(2+k)(1+\mu\beta)}|z|\right\}, \tag{2.18}$$

*and*

$$\left|D_z^{-k} f(z)\right| \leq \frac{|z|^{1+k}}{\Gamma(2+k)} \left\{1 + \frac{(1+\mu)\beta(1-\alpha)}{(2+k)(1+\mu\beta)}|z|\right\}, \tag{2.19}$$

*for $k > 0$ and $z \in U$. The result is sharp.*

**Proof.** This follows directly from taking $\delta = -\rho = -k$ in Theorem 1. Further, equalities are attained for the function $f(z)$ defined by

$$D_z^{-k} f(z) = \frac{z^{1+k}}{\Gamma(2+k)} \left\{1 - \frac{(1+\mu)\beta(1-\alpha)}{(2+k)(1+\mu\beta)}z\right\}, \tag{2.20}$$

or the function $f(z)$ defined by (2.10).

**Corollary 1.** *Under the hypotheses of Theorem 2, $D_z^{-k} f(z)$ is included in the disc with center at the origin and radius $R_1$ given by*

$$R_1 = \frac{1}{\Gamma(2+k)} \left\{1 + \frac{(1+\mu)\beta(1-\alpha)}{(2+k)(1+\mu\beta)}\right\}. \tag{2.21}$$

**Theorem 3.** *Let the function $f(z)$ defined by* (1.2) *be in the class $P^*(\alpha, \beta, \mu)$. Then we have*

$$\left|D_z^{k} f(z)\right| \geq \frac{|z|^{1-k}}{\Gamma(2-k)} \left\{1 - \frac{(1+\mu)\beta(1-\alpha)}{(2-k)(1+\mu\beta)}|z|\right\}, \tag{2.22}$$

*and*

$$\left|D_z^{k} f(z)\right| \leq \frac{|z|^{1-k}}{\Gamma(2-k)} \left\{1 + \frac{(1+\mu)\beta(1-\alpha)}{(2-k)(1+\mu\beta)}|z|\right\}, \tag{2.23}$$

*for* $0 \leq k < 1$ *and* $z \in U$. *The result is sharp.*

**Proof.** Clearly, the proof is obtained by replacing $k$ in Theorem 2 by $-k$. Since the equalities are attained for the function $f(z)$ defined by

$$D_z^k f(z) = \frac{z^{1-k}}{\Gamma(2-k)} \left\{ 1 - \frac{(1+\mu)\beta(1-\alpha)}{(2-k)(1+\mu\beta)} z \right\}, \tag{2.24}$$

or, for the function $f(z)$ defined by (2.10).

**Corollary 2.** *Under the hypotheses of Theorem* 3, $D_z^k f(z)$ *is included in the disk with center at the origin and radius* $R_2$ *given by*

$$R_2 = \frac{1}{\Gamma(2-k)} \left\{ 1 + \frac{(1+\mu)\beta(1-\alpha)}{(2-k)(1+\mu\beta)} \right\}.$$

**Remark.** Putting $\mu = 1$ in Theorems 1, 2 and 3, we get the corresponding results for the class $P^*(\alpha, \beta)$.

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## References

[1] V. P. Gupta and P. K. Jain, *Certain classes of univalent functions with negative coefficients*, Bull. Austral. Math. Soc., 14 (1976), 467-473.

[2] S. Owa, *On the distortion theorem*, I, Kyungpook Math. J., 18 (1978), 53-59.

[3] S. Owa, *Some applications of fractional calculus*, Research Notes in Math. Pitman, Boston, London and Melboume, 138(1985), 164-175.

[4] S. Owa, and M. K. Aouf, *On subclasses of univalent functions with negative coefficients II*, Pure Appl. Math. Soc., 29(1989), 131-139.

[5] H. M. Srivastava and S. Owa (editors), Univalent Functions Fractional Calculus, and Their Applications, Halsted Press(Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1989.

[6] H. M. Srivastava and S. Owa, *An application of the fractional derivative*, Math. Japan., 29(1984), 383-389.

[7] H. M. Srivastava, M. S. Saigo and S. Owa, *A class of distortion theorems involving certain operators of fractional calculus*, J. Math. Anal. Appl., 131 (1988), 412-420.

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