When a skydiver jumps from an airplane, her velocity increases gradually. If she jumps from high enough, however, eventually her velocity approaches a constant value, called the terminal velocity, which is due to air resistance acting on the skydiver. Once the parachute opens, the skydiver slows down rapidly until a new, much smaller, terminal velocity is reached.

People taking skydiving lessons are first trained how to land safely by jumping from a height of about 3.3 m. From this height a freely falling person hits the ground at the same velocity as a skydiver, who falls at a terminal velocity of about 8 m/s once the parachute is opened. If we ignore air resistance, the distance fallen, in metres, after $t$ seconds is $s = 4.9t^2$. We might like to know the velocity of a jumper at any time during the jump. This is given by the derivative of the position function.

We could use the first principles definition of the derivative to compute the velocity. But if we had to do this every time, it would be very time-consuming. Fortunately, mathematicians have come up with several rules to simplify the process of differentiation. In this section, we begin to study these rules.

**Investigate & Inquire: Derivatives of Polynomial Functions**

Many functions that model situations in the sciences or social sciences contain power terms such as $4x^3$. We need to develop an efficient way of differentiating powers in order to find rates of change of those functions. In this investigation, you will look more generally at the derivatives of polynomial functions.

1. Using a graphing calculator, enter the functions shown in the screen to the left below. Graph the functions using a suitable window to get the screen to the right.

   ![Graphing Calculator Screenshots]

   **Window variables:**
   
   $x \in [-4.7, 4.7]$, $y \in [-3.1, 3.1]

   or use the $Z\text{Decimal}$ instruction.
2. The function in Y2 represents the derivative function of Y1. What is the relationship between the original function and the derivative function? Test your hypothesis by entering a different linear function in Y1. Does your hypothesis hold for any linear function?

3. Determine the equation of the derivative function.

4. In the next set of screens, the original function is a quadratic function. What is the relationship between the original function and the derivative function? Test your hypothesis by entering a different quadratic function in Y1. Does your hypothesis hold for any quadratic function?

5. Determine the equation of the derivative function in step 4 and repeat for the other quadratic equations that you have entered.

6. Repeat this process using cubic functions, and then quartic functions. What is the relationship between the original function and the derivative function? Test your hypothesis by entering different cubic and quartic functions in Y1. Does your hypothesis hold?

7. Look back on the equation of each function you explored and the equation of its derivative. Is there a relationship between the equations? Explain.

---

**Power rule:** If \( f(x) = x^n \), where \( n \) is a positive integer, then \( f'(x) = nx^{n-1} \) or \( \frac{d}{dx}x^n = nx^{n-1} \)

You will have the opportunity to derive the **power rule** from first principles in question 25 on page 206.

---

**Example 1  Using the Power Rule**

Differentiate using the power rule.

a) \( f(x) = x^7 \)  

b) \( f(x) = x^{20} \)  

c) \( y = t^4 \)  

d) \( \frac{d}{du}(u^8) \)

**Solution**

a) \( f'(x) = 7x^6 \)  

b) \( f'(x) = 20x^{19} \)  

c) \( y' = 4t^3 \)  

d) \( \frac{d}{du}(u^8) = 8u^7 \)
Note that it is customary to state the derivative in the same notation as the original function. We have stated the power rule when \( n \) is a positive integer, but the rule still applies for any real number \( n \).

**Example 2 Using the Power Rule When \( n \) Is not a Positive Integer**

Use the power rule to find \( f'(2) \) for each function, and then check your result using a graphing calculator or graphing software.

a) \( f(x) = \frac{1}{x^3} \)

b) \( f(x) = \sqrt{x} \)

**Solution**

a) We use a negative exponent to rewrite the function as
\[
f(x) = \frac{1}{x^3} = x^{-3}
\]
Then, the power rule gives
\[
f'(x) = (-3)x^{-3-1} = (-3)x^{-4} = -\frac{3}{x^4}
\]
so
\[
f'(2) = -\frac{3}{(2)^4} = -\frac{3}{16} = -0.1875
\]

b) Here we use a fractional exponent.
\[
y = \sqrt{x} = x^{\frac{1}{2}}
\]
\[
\frac{dy}{dx} = \frac{1}{2} x^{\frac{1}{2}-1} = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}
\]
At \( x = 2 \), \( \frac{dy}{dx} = \frac{1}{2\sqrt{2}} \)

Use the Tangent operation with Window variables:
\( x \in [1, 3], y \in [0, 0.2] \).

Window variables:
\( x \in [0, 9], y \in [0, 3] \)
**Constant rule:** If \( f \) is a constant function, \( f(x) = c \), then \( f'(x) = 0 \) or \( \frac{d}{dx}(c) = 0 \).

The screen illustrates the constant rule geometrically. The graph of a constant function \( f(x) = 3 \) is the horizontal line \( y = 3 \). The tangent at any point on this line is the line itself. Since the horizontal line has slope 0, the slope of the tangent is zero.

**Constant multiple rule:** If \( g(x) = cf(x) \), then \( g'(x) = cf'(x) \) or \( \frac{d}{dx}[cf(x)] = c \frac{d}{dx}f(x) \).

**Sum rule:** If both \( f \) and \( g \) are differentiable, then so is \( f + g \), and \( (f + g)' = f' + g' \) or \( \frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x) \).

**Difference rule:** If both \( f \) and \( g \) are differentiable, then so is \( f - g \), and \( (f - g)' = f' - g' \) or \( \frac{d}{dx}(f(x) - g(x)) = \frac{d}{dx}f(x) - \frac{d}{dx}g(x) \).

---

**Example 3** The Constant Rule and the Constant Multiple Rule

For a parachutist in training from a height of 3.3 m, the distance fallen, \( s \), in metres, after \( t \) seconds is \( s = 4.9t^2 \).

a) State the position function.
b) Find the velocity function.
c) Determine the velocity of the parachutist when she hits the ground.
d) Determine the velocity of the parachutist after 1 s.

**Solution**

a) The position formula, \( s = 4.9t^2 \), is valid only until the parachutist lands. To find the time to land, let \( 4.9t^2 = 3.3 \). Then, \( t = 0.82 \) s

Thus,

\[
s(t) = \begin{cases} 
4.9t^2, & t \in [0, 0.82] \\
3.3, & t \in (0.82, \infty)
\end{cases}
\]

b) The velocity function is

\[
v(t) = s'(t) = \begin{cases} 
9.8t, & t \in [0, 0.82] \\
0, & t \in (0.82, \infty)
\end{cases}
\]

c) \( v(0.82) = 9.8(0.82) = 8.036 \)

The velocity of the parachutist when she hits the ground is 8.036 m/s.
d) \( v(1) = 0 \)

The velocity after 1 s is 0 m/s; after the parachutist hits the ground, her speed is 0.
Example 4 Using Derivative Rules

Find the derivative of each function.

a) \[ y = 3 \quad \text{b) } f(x) = 8x^{\frac{3}{2}} \quad \text{c) } f(x) = 2x^2 - 3x + 4 \quad \text{d) } g(x) = 5x^3 - \sqrt{x} \]

Solution

a) We use the constant rule.
If \( y = 3 \), then \( \frac{dy}{dx} = 0 \).

b) Using the constant multiple rule, we get
\[ f(x) = 8x^{\frac{3}{2}} \]
\[ f'(x) = 8 \cdot \frac{d}{dx} \left( x^{\frac{3}{2}} \right) \]
\[ = 8 \left( \frac{3}{2} x^{\frac{1}{2}} \right) \]
\[ = 12x^{\frac{1}{2}} \]

c) Combining the sum and difference rules with the power rule and the constant multiple rule, we obtain
\[ f'(x) = \frac{d}{dx} (2x^2 - 3x + 4) \]
\[ = 2 \cdot \frac{d}{dx} (x^2) - 3 \cdot \frac{d}{dx} (x) + \frac{d}{dx} (4) \]
\[ = 2(2x) - 3(1) + 0 \]
\[ = 4x - 3 \]

d) We use the difference rule, the constant multiple rule, and the power rule:
\[ g'(x) = \frac{d}{dx} (5x^3 - \sqrt{x}) \]
\[ = 5 \cdot \frac{d}{dx} (x^3) - \frac{d}{dx} \left( x^{\frac{1}{2}} \right) \]
\[ = 5(3x^2) - \frac{1}{2} x^{-\frac{1}{2}} \]
\[ = 15x^2 - \frac{1}{2} x^{-\frac{1}{2}} \]
\[ = 15x^2 - \frac{1}{2\sqrt{x}} \]

Example 5 Equation of a Tangent

Find the equation of the tangent to the curve \( y = \frac{1}{x} \) at \( x = -2 \). Check your result using a graphing calculator or graphing software.

Solution

The curve is the graph of the function \( f(x) = x^{-1} \).

Window variables:
\( x \in [-4.7, 4.7], y \in [-3.1, 3.1] \)

We know that the slope of the tangent at \( x = -2 \) is the derivative evaluated at \(-2\), that is, \( f'(-2) \).
From the power rule, \\
\( f'(x) = -x^2 \) \\
Thus, \( f'(-2) = -(-2)^2 = -\frac{1}{4} \) \\
Also, \\
\( f(-2) = \frac{1}{2} \) \\
\[ = -\frac{1}{2} \]
To find the equation of the tangent at \( x = -2 \), use the point-slope form \\
\[ y - y_1 = m(x - x_1) \]
\[ y - \left( -\frac{1}{2} \right) = -\frac{1}{4}(x - (-2)) \]
\[ y = -\frac{1}{4}x - 1 \]
The equation of the tangent line at \( x = -2 \) is \( y = -\frac{1}{4}x - 1 \). \\
Use the **Tangent operation** to check this equation.

---

**Example 6  Equations of Tangents**

Find the equations of the tangents to the parabola \( y = x(x + 2) \) that pass through the point \((1, -6)\). Sketch the curve and the tangents.

**Solution**

Initially, we expand the brackets so that the equation is written \( y = x^2 + 2x \). 
(In Section 4.3, we develop a product rule, which will allow us to differentiate without first expanding.)

Let the \( x \)-coordinate of the point \( Q \), where the tangent touches the parabola, be \( a \). 
Then, the coordinates of the point \( Q \) are \((a, a^2 + 2a)\). To determine the values of \( a \), 
we express the slope of the tangent \( PQ \) in two ways. Using the formula for slope, 
we have 
\[ m_{PQ} = \frac{y_2 - y_1}{x_2 - x_1} \]
\[ = \frac{(a^2 + 2a) - (-6)}{a - 1} \]
Also, we know that the slope of the tangent at \( Q \) is \( f'(a) \), where \( f(x) = x^2 + 2x \). Thus 
\( f'(x) = 2x + 2 \) 
so the equation \( m_{PQ} = f'(a) \) becomes 
\[ \frac{a^2 + 2a + 6}{a - 1} = 2a + 2 \]
Now we solve for \( a \) to obtain
\[
\begin{align*}
a^2 + 2a + 6 &= (a - 1)(2a + 2) \\
a^2 + 2a + 6 &= 2a^2 - 2 \\
a^2 - 2a - 8 &= 0 \\
(a - 4)(a + 2) &= 0 \\
a &= 4 \text{ or } -2
\end{align*}
\]
By substitution,
\[
\begin{align*}
f(4) &= (4)^2 + 2(4) \\
&= 24 \\
f(-2) &= (-2)^2 + 2(-2) \\
&= 0
\end{align*}
\]
The points of contact are (4, 24) and (-2, 0). The slopes of the tangents at these points are
\[
\begin{align*}
f'(4) &= 2(4) + 2 \\
&= 10
\end{align*}
\]
and
\[
\begin{align*}
f'(-2) &= 2(-2) + 2 \\
&= -2
\end{align*}
\]
The equations of the tangents at (4, 24) and (-2, 0) are
\[
\begin{align*}
y - 24 &= 10(x - 4) \quad \text{and} \quad y - 0 &= -2(x - (-2)) \\
y &= 10x - 16 \quad \text{and} \quad y &= -2x - 4
\end{align*}
\]

Key Concepts

The basic derivative rules:

<table>
<thead>
<tr>
<th>Rule</th>
<th>Function Notation</th>
<th>Leibniz Notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Power</td>
<td>If ( f(x) = x^n ), then ( f'(x) = nx^{n-1} ).</td>
<td>( \frac{d}{dx} x^n = nx^{n-1} )</td>
</tr>
<tr>
<td>Constant</td>
<td>If ( f ) is a constant function, ( f(x) = c ), then ( f'(x) = 0 ).</td>
<td>( \frac{d}{dx} c = 0 )</td>
</tr>
<tr>
<td>Constant multiple</td>
<td>If ( g(x) = cf(x) ), then ( g'(x) = cf'(x) ).</td>
<td>( \frac{d}{dx} [cf(x)] = c \frac{d}{dx} f(x) )</td>
</tr>
<tr>
<td>Sum</td>
<td>If both ( f ) and ( g ) are differentiable, then so is ( f + g ), and ( (f + g)' = f' + g' ).</td>
<td>( \frac{d}{dx} (f(x) + g(x)) = \frac{d}{dx} f(x) + \frac{d}{dx} g(x) )</td>
</tr>
<tr>
<td>Difference</td>
<td>If both ( f ) and ( g ) are differentiable, then so is ( f - g ), and ( (f - g)' = f' - g' ).</td>
<td>( \frac{d}{dx} (f(x) - g(x)) = \frac{d}{dx} f(x) - \frac{d}{dx} g(x) )</td>
</tr>
</tbody>
</table>

Communicate Your Understanding

1. Give a geometric explanation for each of the following.
   a) The derivative of a constant function is zero.
   b) The derivative of a linear function is a constant.
2. Describe how you would find an equation of the tangent to a function of the form \( y = cx^n \) at \( x = a \).

3. Problems involving tangents come in three forms: finding the tangent equation given a point on the curve; finding the tangent equation, given the slope; or finding the equation of a tangent that passes through a point not on the curve.
   a) Describe how to identify which type of problem is being posed.
   b) Explain how to solve each type of problem.

4. Describe how to find the derivative of the sum or difference of two differentiable functions.

5. Describe the steps you would use to find the derivative of \( y = (ax + b)^3 \).
9. Find the equation of the tangent to the curve at the given point.
   a) \( y = x^3 - 3x^2 + x + 3 \) at \((-1, -2)\)
   b) \( y = x^2 - 4\sqrt{x} \) at \((4, 8)\)
   c) \( y = \frac{x^4 - 6x^2}{3x} \) at \((3, 3)\)
   d) \( y = -4 + \frac{4}{x} - \frac{8}{x^2} \) at \((2, -4)\)
   e) \( y = (x^2 - 3)^2 \) at \((-2, 1)\)
   f) \( y = (x + 1)^2 \) at \((1, 4)\)

Apply, Solve, Communicate

10. At what point on the parabola \( y = 4x^2 \) is the slope of the tangent equal to 24?

11. Find the points on the curve \( y = 2 - \frac{1}{x} \) where the tangent is perpendicular to the line \( y = 1 - 4x \).

12. a) Use a graphing calculator or graphing software to graph \( y = x^2 \) and two tangents, the first at \( x = -2 \) and the other at \( x = 2 \). Where do the two tangents intersect?
   b) There are two tangents to \( y = x^2 \) that pass through the point \((0, -5)\). Find the coordinates of the points where these tangents meet the parabola.

13. Inquiry/Problem Solving The projection for the gain in net worth, in dollars, of a new technology company is given by the equation, \( N(t) = 50000t^2 \), where \( t \) is in years.
   a) According to this projection, at what rate will the net worth of the company be increasing with respect to time after 7 years?
   b) Draw the graph of the function using a graphing calculator or graphing software. If you could invest money in this company only for a short period of time, when would be the best time to invest? Explain.

14. If a ball is thrown upward with a velocity of 30 m/s, its height, \( h \), in metres, after \( t \) seconds is given by \( h = 1 + 30t - 4.9t^2 \). Find the velocity of the ball after 1 s, 3 s, and 5 s.

15. Communication Many materials, such as metals, form an oxide coating (rust) on their surfaces that increases in thickness, \( x \), in centimetres, over time, \( t \), in years, according to the equation \( x = kt^{\frac{1}{2}} \).
   a) Find the growth rate, \( G = \frac{dx}{dt} \), as a function of time.
   b) If \( k = 0.02 \), find the growth rate after 4 years.
   c) Using a graphing calculator or graphing software, graph both \( x \) and \( G \) for \( k = 0.02 \). What happens to the thickness and the growth rate as \( t \) increases?
   d) Why is it very important to protect the new metal on cars from rusting (oxidizing) as soon as possible?

16. The position, \( s \), in metres, of an accelerating car on a highway is given by \( s = 20 + 5t + 0.5t^2 \), where \( t \) is measured in seconds. Find the velocity of the car at 4 s, 6 s, and 10 s.

17. Inquiry/Problem Solving A car whose position is given by the equation \( d = 25t + t^2 \) passes a police car that is travelling at 20 m/s. The police officer turns on the siren and begins to accelerate at 1.5 m/s\(^2\) to chase the speeding car.
   a) At what time is the speeding car moving at 31 m/s?
   b) How fast is the police car moving at that time?
   c) How far apart are the vehicles at that time?
   d) Will the police car catch up to the speeder? Explain.

18. Application At what points does the curve \( y = 2x^3 + 3x^2 - 36x + 40 \) have a horizontal tangent?

19. Show that the curve \( y = 2x^3 + 3x - 4 \) has no tangents with slope 2.

20. Inquiry/Problem Solving Find the equations of both lines that pass through the origin and are tangent to the parabola \( y = x^2 + 4 \).

21. Find the x-coordinates of the points on the curve \( xy = 1 \) where the tangents from the point \((1, -1)\) intersect the curve.
22. The cost, \( C \), in dollars, for a company to produce \( x \) copies of a videotape is given by 
\[
C(x) = 100000 + 0.1x + 0.01x^2, \quad x \in [0, 1000]
\]
   a) Find \( C'(x) \) and \( C'(101) \).
   b) Find the cost of producing the 101st copy of the video using \( C(x) \).
   c) What does \( C'(x) \) represent? Compare \( C'(101) \) and your result for part b).

23. Application An oil tanker is leaking oil into the ocean on a calm day when there are few waves. The oil is spreading out in a circular pattern on the surface of the water. The area of the circular oil spill is given by the formula \( A = \pi r^2 \).
   a) Find the area in terms of the diameter.
   b) Find the area of the circle when the rate of change of the area with respect to the diameter is 400 square metres per metre.

24. Communication The cost of manufacturing \( x \) units of a product is \( C(x) = 5\sqrt{x} \) dollars, ignoring any fixed costs.
   a) Find \( \frac{dC}{dx} \) and explain what it represents.
   b) Graph \( \frac{dC}{dx} \) using a graphing calculator or graphing software. What happens to the graph as \( x \) increases? Explain why this might occur when a company produces more units of a product.

25. a) Show that, if \( f(x) \) is a polynomial function, the quotient \( Q(x) = \frac{f(x) - f(a)}{x - a} \) is also a polynomial function.
   b) Let \( f(x) = x^n \), where \( n \) is a positive integer. Write a formula for \( Q(x) \).
   c) Use part b) to derive the power rule from first principles.

26. Communication The volume of a blood vessel of length \( h \) and radius \( r \) can be approximated by using the formula for the volume of a cylinder, 
\[ V = \pi r^2 h \]
   a) Over time the radius of the inside of the blood vessel can decrease due to plaque deposits on the inner wall of the blood vessel. Typically, the length of the blood vessel does not change in these cases. Find \( \frac{dV}{dr} \) and explain its meaning.
   b) In some cases the length of the blood vessel may change, especially in young children who are growing taller. Typically, the radius does not change when the person is growing. Find \( \frac{dV}{dh} \) and explain what it represents.

27. a) Prove the sum rule using the definition of the derivative.
   b) Prove the difference rule.
   c) Prove the constant multiple rule.
   d) Use realistic examples of rates of change to explain why the rules of parts a) to c) are reasonable.

28. Give a geometric explanation for each of the following.
   a) The derivative of \( y = cx^n \) is greater than the derivative of \( y = x^n \) if \( c > 1 \) and \( x > 0 \).
   b) The derivative of \( y = cx^n \) is less than the derivative of \( y = x^n \) if \( 0 < c < 1 \) and \( x > 0 \).

29. Let
\[
f(x) = \begin{cases} 
3 - 2x & \text{if } x \in (-\infty, -1) \\
x^2 + 4 & \text{if } x \in [-1, 1] \\
2x + 3 & \text{if } x \in (1, \infty) 
\end{cases}
\]
   a) Sketch the graph of \( f \).
   b) Where is \( f \) differentiable? Explain.
   c) Find an expression for \( f' \) and sketch the graph of \( f' \).

30. a) Sketch the graph of \( f(x) = |x^2 - 9| \).
   b) For what values of \( x \) is \( f \) not differentiable?
   c) Find a formula for \( f' \) and sketch the graph.

31. A man is walking on a bus with position function \( s(t) = 5t \) with respect to the bus, where \( s \) is measured in metres and \( t \), in seconds. The bus has position function \( r(t) = 0.02t^2 \) with respect to the ground.
   a) The velocity of the man if he is walking toward the back of the bus. Explain your reasoning.
   b) The velocity of the man if he is walking toward the front of the bus. Explain your reasoning.
The length, $L$, in millimetres, of a column of liquid inside a thermometer, as a function of temperature, $T$, in degrees Celsius, is given by the equation

$$L = L_0(1 + T + 0.002T^2)$$

For a certain thermometer, let $L_0 = 30$.

a) What is the length of the column of liquid at the following temperatures?
   i) $0^\circ$  ii) $10^\circ$  iii) $20^\circ$  iv) $30^\circ$

b) When designers construct the thermometers, they often assume the relationship is a linear one (ignoring the last term in the equation), and they make the temperature divisions evenly spaced. Where will the manufacturer place the divisions to indicate the temperatures in part a)?

c) Calculate the error for each temperature in part a) if the approximation in part b) is made. What happens to the error as the temperature increases?

d) Find $\frac{dL}{dT}$ and sketch its graph. What does $\frac{dL}{dT}$ represent?

e) Use your result in part d) to explain the error in the temperature readings.