The expansion of a semigroup and a Riesz basis criterion

Gen Qi Xu\(^a\),\(^*\), Siu Pang Yung\(^b\)

\(^a\) Mathematics Department, Tianjin University, Tianjin, 300072, PR China
\(^b\) Mathematics Department, University of Hong Kong, HongKong, PR China

Received 5 November 2002; revised 29 November 2003
Available online 16 December 2004

Abstract

Problems on the expansion of a semigroup and a criterion for being a Riesz basis are discussed in the present paper. Suppose that \(A\) is the generator of a \(C_0\) semigroup on a Hilbert space and \(\sigma(A) = \sigma_1(A) \cup \sigma_2(A)\) with \(\sigma_2(A)\) is consisted of isolated eigenvalues distributed in a vertical strip. It is proved that if \(\sigma_2(A)\) is separated and for each \(\lambda \in \sigma_2(A)\), the dimension of its root subspace is uniformly bounded, then the generalized eigenvectors associated with \(\sigma_2(A)\) form an \(L^2\)-basis. Under different conditions on the Riesz projection, the expansion of a semigroup is studied. In particular, a simple criterion for the generalized eigenvectors forming a Riesz basis is given. As an application, a heat exchanger problem with boundary feedback is investigated. It is proved that the heat exchanger system is a Riesz system in a suitable state Hilbert space.

© 2004 Elsevier Inc. All rights reserved.

Keywords: Semigroup expansion; Riesz basis; Heat exchanger equation

\(^*\) Corresponding author.
E-mail addresses: gqxu@mail.sxu.edu.cn (G.Q. Xu), spyung@hku.hk (S.P. Yung).
1. Introduction

Let $X$ be a Banach space and $\mathcal{A}$ be a densely defined and closed linear operator in $X$. We consider the evolutionary equation in $X$

$$\begin{cases}
    \frac{d}{dt} \Phi(t) = \mathcal{A} \Phi(t), & t > 0, \\
    \Phi(0) = \phi_0.
\end{cases}$$

(1.1)

Suppose that $\mathcal{A}$ generates a $C_0$ semigroup $T(t)$ on $X$. Then the solution of (1.1) can be written into

$$\Phi(t) = T(t) \phi_0.$$  

In particular, if the spectrum of $\mathcal{A}$ is of the form $\sigma(\mathcal{A}) = \sigma_1(\mathcal{A}) \cup \sigma_2(\mathcal{A})$ where $\sigma_2(\mathcal{A}) = \{ \lambda_k \}_{k=1}^r$ is consisted of isolated eigenvalues of $\mathcal{A}$ with finite multiplicity, then when $r < \infty$, the solution to (1.1) has the form

$$\Phi(t) = \sum_{k=1}^r e^{\lambda_k t} P_k(t, \phi_0) + R(t, \phi_0),$$

(1.2)

where $P_k(t, \phi_0)$ are $X$-valued polynomials in $t$.

When $r = \infty$, it is natural to ask whether equality (1.2) still remains true. When $\mathcal{A}$ has compact resolvent, the question then becomes whether the residual term $R(t, \phi_0)$ vanishes. If $\mathcal{A}$ is a non-positive self-adjoint or, more generally, a skew-adjoint operator, the answer is clear. If $\mathcal{A}$ is a non-self-adjoint, or in general non-normal operator (e.g. transport operator [17]), then the answer is not so clear and answering this question is important in applications.

In the control theory of distributed parameter systems, when a boundary feedback control acts on a vibration system, it usually yields a non-self-adjoint operator $\mathcal{A}$ which satisfies the following properties: (1) $\mathcal{A}$ has compact resolvents and generates a $C_0$ semigroup; (2) the spectrum $\sigma(\mathcal{A})$ lies inside some vertical strip; (3) the eigenvalues of $\mathcal{A}$ are separated and the algebraic multiplicity of the eigenvalues are bounded uniformly. There are many examples in practice are of this type. For instance, strings [7,24], Euler–Bernoulli beams [8,13,14], Rayleigh beams [22], Timoshenko beams [25,29], heat exchanger [16], etc. For these type of problems, to investigate the expansion issue of (1.2), we plan to do two things:

(I) one is to show that there exists a direct sum decomposition of $X$:

$$X = X_1 + X_2,$$

(1.3)

where $X_1$ and $X_2$ are $T(t)$-invariant closed subspaces such that $\sigma(\mathcal{A}|_{X_1}) = \sigma_1(\mathcal{A})$ and $\sigma(\mathcal{A}|_{X_2}) = \sigma_2(\mathcal{A})$;
(II) another is to show that, for each \( \phi_0 \in X_2 \), the non-harmonic series
\[
\sum_{k=1}^{\infty} e^{\lambda_k t} P_k(t, \phi_0)
\] (1.4)
converges in some sense.

If these two assertions hold, then (1.2) holds true for \( r = \infty \). However, difficulties occur because (I) concerns with the splitting of the spectrum for unbounded linear operators, and (II) concerns with establishing the basis property.

Assume now that \( X \) is separable, \( A \) has a compact resolvent and the algebraic multiplicities of the eigenvalues of \( A \) are uniformly bounded. We say that \( A \) is a spectral operator if there is a sequence of generalized eigenvectors of \( A \) that forms a unconditional basis for \( X \) (see, e.g. [10]). In this case, the system (1.1) is also said to be a spectral system. If \( X \) is a Hilbert space, it would be called a Riesz system. A spectral system will always satisfy the spectrum determined growth assumption, so the stability of the system can be determined quite conveniently via the spectrum of \( A \). The usual way in the aforementioned references in proving (II) is to asymptotically estimate the eigenvalues and eigenvectors, and then deduces that the system is a Riesz one. This approach seems unsuitable here for the convergence of the series (1.4) in our case because (1.4) involves the family of exponentials
\[
\mathcal{E}^{(k)} = \{e^{\lambda_k t}, te^{\lambda_k t}, \ldots, t^{m_k-1}e^{\lambda_k t}\}, \quad k \geq 1,
\] (1.5)
with \( m_k \) being the chain length of \( \lambda_k \) (or the ascent of operator \( (\lambda_k I - A) \)). For references on the family of exponentials, see for instance [2–6,18,19] and references therein.

In this paper, we shall prove the convergence of the series (1.4) by establishing some property for the family of exponentials. The main result of this paper is as follows.

**Theorem 1.1.** Let \( X \) be a separable Hilbert space, and \( A \) be the generator of a \( C_0 \) semigroup \( T(t) \) on \( X \). Suppose that:

1. \( \sigma(A) = \sigma_1(A) \cup \sigma_2(A) \) where \( \sigma_2(A) = \{\lambda_k\}_{k=1}^{\infty} \) consists of isolated eigenvalues of \( A \) with finite multiplicity;
2. \( \sup m_\alpha(\lambda_k) < \infty \), where \( m_\alpha(\lambda_k) = \dim E(\lambda_k, A)X \) and \( E(\lambda_k, A) \) is the Riesz projector associated with \( \lambda_k \);
3. there is a constant \( z \) such that
\[
\sup \{\Re \lambda \mid \lambda \in \sigma_1(A)\} \leq z \leq \inf \{\Re \lambda \mid \lambda \in \sigma_2(A)\}
\]
and
\[
\inf_{n \neq m} |\lambda_n - \lambda_m| > 0.
\] (1.6)
Then the following assertions are true.

(i) There exist two $T(t)$-invariant closed subspaces $X_1$ and $X_2$ with the property that

\[ \sigma(A|_{X_1}) = \sigma_1(A), \quad \sigma(A|_{X_2}) = \sigma_2(A), \quad \{E(\lambda_k, A)X_2\}_{k=1}^{\infty} \]

forms a subspace Riesz basis for $X_2$ (the definition of subspace Riesz basis can be found in [12]), and

\[ X = X_1 \oplus X_2. \]

(ii) If \( \sup_{k \geq 1} ||E(\lambda_k, A)|| < \infty \), then

\[ \mathcal{D}(A) \subset X_1 \oplus X_2 \subset X. \]

(iii) $X$ has the decomposition

\[ X = X_1 \oplus X_2 \quad \text{(topological direct sum)} \]

if and only if \( \sup_{n \geq 1} \left| \sum_{k=1}^{n} E(\lambda_k, A) \right| < \infty \).

In Theorem 1.1, the first assertion says that for each $\phi_0 \in X_1 \oplus X_2$, the solution of (1.1) can be expressed in terms of the generalized eigenvectors. The second assertion says that the same is true for all classical solutions of (1.1) if $\sup_{k} ||E(\lambda_k, A)|| < \infty$. The third assertion is a sufficient and necessary condition for the semigroup expansion. In particular, if $\sigma_1(A) = \emptyset$ and $X_2 = X$, Theorem 1.1 concludes that $\{E(\lambda_k, A)X; \ k \geq 1\}$ forms a subspace Riesz basis in $X$.

As a consequence of above theorem, we have the following corollary.

**Corollary 1.1.** Let $X$ be a separable Hilbert space and assume that $A$, the generator a $C_0$ semigroup $T(t)$ on $X$, have compact resolvents and generate a $C_0$ semigroup $T(t)$ on $X$. Suppose that the dimensions of all eigenspaces of $A$ are bounded uniformly and the resolvent of $A$ can be expressed as

\[ R(\lambda, A)F = \frac{G(\lambda, F)}{\Gamma(\lambda)} \]

with $G(\lambda, F)$ being some entire function valued in $X$. Assume that $\Gamma(z)$ satisfies the following properties:

(1) $\Gamma(z)$ is an analytic function on the complex plane, and the set of zeros of $\Gamma(z)$ coincides with $\sigma(A)$;

(2) there exist constants $\varepsilon > 0$ and $N > 0$ such that as $|\text{Re} \ z| \geq N$ we have $|\Gamma(z)| \geq \varepsilon$;

(3) when $|\text{Re} \ z| \leq N$ and $|\text{Im} \ z|$ is large enough, $\Gamma(z)$ has an asymptotic expression

\[ \Gamma(z) = G(z) + R(z), \]
where \( R(z) = O(z^{-1}) \) and \( G(iz) \) is an entire function of sine type in the sense of Young's book [30].

Then the system of generalized eigenvectors of \( A \) forms a Riesz basis in \( Sp(\mathcal{A}) \) where

\[
Sp(\mathcal{A}) := \left\{ \sum_{k=1}^{m} \sum_{j=1}^{n_k} a_{k,j} \phi_{k,j}, \left| a_{k,j} \in \mathbb{C}, \phi_{k,j} \text{ are the generalized eigenvectors of } A \right|, m \geq 1 \right\}.
\]

In particular, if the system of generalized eigenvectors of \( A \) is complete in \( X \), i.e., \( Sp(\mathcal{A}) = X \), then it forms a Riesz basis in \( X \).

It is a useful proposition because its conditions are easily verified in practical problems. For instance, if \( \mathcal{A} \) is determined by an ordinary differential operator with strongly regular boundary condition in the sense of Naimark (see [20]), then all conditions of Corollary 1.1 are fulfilled, and hence \( \mathcal{A} \) is a spectral operator. For related development on ordinary differential operators, we refer to [1,23,26].

The present paper is arranged as follows. In Section 2, we shall prove the main result and its corollary. In Section 3, we shall apply the results to an example: heat exchanger with boundary feedback controls. The only assumption that we need is the gain coefficients being non-zero, and we show that the closed loop heat exchanger system is a Riesz one in the corresponding state Hilbert space.

2. The proofs of the main result and its corollary

We begin with some notations and lemmas.

Let \( \lambda_n; n \in \mathbb{N} \) be a complex sequence ordered in such a way that \( \{\text{Im} \lambda_n; n \in \mathbb{N}\} \) forms a non-decreasing sequence. In what follows, we always assume that \( \sup_n |\text{Re} \lambda_n| < \infty \).

The set \( \Lambda = \{\lambda_n; n \in \mathbb{N}\} \) is said to be separated if the condition

\[
\delta(\Lambda) := \inf_{k \neq n} |\lambda_k - \lambda_n| > 0
\]

is fulfilled.

Here, we make a distinction between the separability of a set and that of a sequence. If a sequence is separated, all points \( \lambda_n \) are entirely distinct. When a set is separated, its elements are allowed to repeat themselves. In this case, we will denote by \( m_n \) the number of times repeated and called it the multiplicity of the element \( \lambda_n \in \Lambda \).

To each \( \lambda_n \in \Lambda \), we associate it with an exponential family

\[
\mathcal{E}^{(n)}(\Lambda) = \{e^{\lambda_n t}, t e^{\lambda_n t}, \ldots, t^{m_n-1} e^{\lambda_n t}\}, \quad n \geq 1
\]

and \( m_n \) is the multiplicity of \( \lambda_n \in \Lambda \).
We say that the family of exponentials, \( \{ e^{nAf} \} \), is an L-basis in \( L^2[0, T] \) if it is a Riesz basis in the subspace

\[
U = \text{span} \left\{ \sum_{n=1}^{N} \sum_{j=0}^{m_n-1} a_{n,j} t^j e^{\lambda_n t} \mid \forall N \geq 1 \right\}.
\]

(2.3)

Denote by \#G the number of elements in the sequence G (or the set G, taking the multiplicity into account in this case), and put

\[
n^+(r) := \sup_{x \in \mathbb{R}} \#(\text{Im} A \cap [x, x + r]), \quad n^-(r) := \inf_{x \in \mathbb{R}} \#(\text{Im} A \cap [x, x + r]).
\]

Define the upper \( D^+(A) \) and the lower \( D^-(A) \) uniform densities of \( A \) be respectively

\[
D^+(A) := \lim_{r \to \infty} \frac{n^+(r)}{r}, \quad D^-(A) := \lim_{r \to \infty} \frac{n^-(r)}{r}.
\]

Both the limits exist due to the subadditivity of \( n^+(r) \) and the superadditivity of \( n^-(r) \).

**Lemma 2.1** (Avdonin et al., 2001 [3, Theorem 3]). Let \( A = \{ \lambda_n \} \) be a sequence with property

\[
\sup_{n \geq 1} |\text{Re} \lambda_n| < \infty.
\]

If \( A \) is separated as a set and \( \sup_n m_n < \infty \), then the following statements are valid.

1. For any \( T < 2\pi D^-(A) \), there exists a subfamily \( E_0 \) of \( \{ E^{(n)}(A) \} \) that forms a Riesz basis in \( L^2(0, T) \), and the family \( \{ E^{(n)}(A) \} \) has infinite elements.
2. For any \( T > 2\pi D^+(A) \), the family \( \{ E^{(n)}(A) \} \) forms an L-basis in \( L^2(0, T) \), and it can be extended to a family \( E_1 \) that forms a Riesz basis in this space. The family \( E_1 \setminus \{ E^{(n)}(A) \} \) also contains infinite elements.

**Lemma 2.2** (Curtain and Zwart [9, Lemma 2.5.6, p. 70]). Let \( T(t) \) be a \( C_0 \) semigroup on Hilbert space Z with infinitesimal generator \( \mathcal{A} \). Let \( \rho_{\infty}(\mathcal{A}) \) denote the (maximal) component of the resolvent set \( \rho(\mathcal{A}) \) that contains an interval \( [r, \infty) \). For the closed subspace V, the following statements are equivalent:

1. \( V \) is \( T(t) \)-invariant,
2. \( V \) is \( (\lambda I - \mathcal{A})^{-1} \)-invariant for one \( \lambda \) in \( \rho_{\infty}(\mathcal{A}) \),
3. \( V \) is \( (\lambda I - \mathcal{A})^{-1} \)-invariant for all \( \lambda \) in \( \rho_{\infty}(\mathcal{A}) \).
Lemma 2.3 (Curtain and Zwart [9, Lemma 2.5.3, p. 69]). Let $V$ be a closed subspace of Hilbert space $Z$ and let $\mathcal{A}$ be the infinitesimal generator of a $C_0$ semigroup $T(t)$ on $Z$. If $V$ is $T(t)$-invariant, then the following assertions hold.

1. $V$ is $\mathcal{A}$-invariant.
2. $T(t)|_V$ is a $C_0$ semigroup on $V$ with infinitesimal generator $\mathcal{A}_v$, where $\mathcal{A}_v g = \mathcal{A} g$ for $g \in \mathcal{D}(\mathcal{A}_v) = \mathcal{D}(\mathcal{A}) \cap V$.

Proof of Theorem 1.1. Let $X$ be a separable Hilbert space, and let $\mathcal{A}$ be the generator of a $C_0$ semigroup $T(t)$ on $X$ and satisfy conditions:

1. $\sigma(\mathcal{A}) = \sigma_1(\mathcal{A}) \cup \sigma_2(\mathcal{A})$ with $\sigma_2(\mathcal{A}) = \{\lambda_k\}_{k=1}^{\infty}$ consists of isolated eigenvalues of $\mathcal{A}$ with finite multiplicity;
2. $\sup_{k \geq 1} m_\sigma(\lambda_k) < \infty$; \hspace{1cm} (2.4)
3. $\sup \{\Re \lambda \mid \lambda \in \sigma_1(\mathcal{A})\} \leq \alpha \leq \inf \{\Re \lambda \mid \lambda \in \sigma_2(\mathcal{A})\}$ \hspace{1cm} (2.5)

for some constant $\alpha$ and $\sigma_2(\mathcal{A})$ is a separated set, that is

$$\inf_{n \neq m} |\lambda_n - \lambda_m| > 0$$ \hspace{1cm} (2.6)

for $\lambda_n, \lambda_m \in \sigma_2(\mathcal{A})$.

Now we define the set

$$Sp_{\sigma_2}(\mathcal{A}) := \left\{ \sum_{k=1}^{m} E(\lambda_k, \mathcal{A}) \phi \mid \forall \phi \in X \quad \forall m \in \mathbb{N} \right\}$$ \hspace{1cm} (2.7)

and the closed subspace

$$X_2 := \overline{Sp_{\sigma_2}(\mathcal{A})}.$$ \hspace{1cm} (2.8)

It is easy to see that $X_2$ is a $T(t)$-invariant closed subspace, and hence $\mathcal{A}$-invariant (see Lemma 2.3). Conditions (2.4)–(2.6) imply that the set $\sigma_2(\mathcal{A})$ satisfies all conditions in Lemma 2.1. Thus, the family $\{\mathcal{E}^{(k)}(\sigma_2), k \in \mathbb{N}\}$ forms a $\mathcal{L}$-basis in $L^2[0, T]$ for $T > 2\pi D^+(\sigma_2)$. 
We complete the proof of Theorem 1.1 in four steps.

Step 1: \( \{E(\lambda_k, A)X_2, k \geq 1\} \) forms a subspace Riesz basis in \( X_2 \).

For any \( f \in X \) and \( \phi \in \text{Sp}_{\sigma_2}(A) \), the function \( (T(t)\phi, f) \) is in the subspace \( U \) (the definition see (2.3)). Then, for any \( \phi \in X_2 \), \( (T(t)\phi, f) \in U \) and

\[
(T(t)\phi, f) = \sum_{k=1}^{\infty} \sum_{j=0}^{m_k} \frac{t^j}{j!} e^{\lambda_k t} ((A - \lambda_k I)^j E(\lambda_k, A)\phi, f).
\]

From the Riesz basis property of \( \{E^{(n)}(\sigma_2)\} \) in \( U \), we know that there exist positive constants \( C_1 \) and \( C_2 \) such that

\[
C_1 \sum_{k=1}^{\infty} \sum_{j=0}^{m_k} \left| \frac{((A - \lambda_k I)^j E(\lambda_k, A)\phi, f)}{j!} \right|^2 
\leq \int_0^T \left| (T(t)\phi, f) \right|^2 dt 
\leq C_2 \sum_{k=1}^{\infty} \sum_{j=0}^{m_k} \left| \frac{(A - \lambda_k I)^j E(\lambda_k, A)\phi, f}{j!} \right|^2.
\]

Let \( T(t) \) satisfy \( ||T(t)|| \leq Me^{\omega t} \). Then we get

\[
C_1 \sum_{k=1}^{\infty} \sum_{j=0}^{m_k} \left| \frac{((A - \lambda_k I)^j E(\lambda_k, A)\phi, f)}{j!} \right|^2 \leq M \frac{e^{2\omega T} - 1}{2\omega} ||\phi||^2 ||f||^2 \tag{2.9}
\]

which implies

\[
C_1 \sum_{k=1}^{\infty} |(E(\lambda_k, A)\phi, f)|^2 \leq M \frac{e^{2\omega T} - 1}{2\omega} ||\phi||^2 ||f||^2.
\]

Consequently,

\[
C_1 \sum_{k=1}^{\infty} ||(E(\lambda_k, A)\phi)||^2 \leq M \frac{e^{2\omega T} - 1}{2\omega} ||\phi||^2 \tag{2.10}
\]
According to Lemmas 2.2 and 2.3, \( T(t) \) is a \( C_0 \) semigroup in \( X_2 \) and its generator is \( A|_{X_2} \) with domain \( D = D(A) \cap X_2 \). Lemma 2.2 shows that each \( \lambda_k \) is an isolated eigenvalue of \( A|_{X_2} \) with finite multiplicity. Let \( A^\dagger \) and \( T^\dagger(t) \) be the adjoint operator of \( A \) and \( T(t) \) restricted to \( X_2 \), respectively. Since \( X_2 \) endowed with \( ||\cdot||_{X} \) is also a Hilbert space, \( T^\dagger(t) \) is a \( C_0 \) semigroup and its generator is \( A^\dagger \). Then \( A^\dagger \) has the property that each \( \lambda_k \) is an isolated eigenvalue of \( A^\dagger \) with finite multiplicity, and \( E^\dagger(\lambda_k, A) = E(\lambda_k, A^\dagger) \).

For any \( f, g \in X_2 \), \( (T^\dagger(t)g, f) = (g, T(t)f) \in U \) and

\[
(T^\dagger(t)\phi, f) = \sum_{k=1}^{\infty} \sum_{j=0}^{m_k} \frac{t^j}{j!} e^{\lambda_k t} ((A^\dagger - \lambda_k I)^j E(\lambda_k, A^\dagger)g, f).
\]

With an entirely parallel approach, we can prove that

\[
C_1 \sum_{k=1}^{\infty} ||E^\dagger(\lambda_k, A)g||_{X_2}^2 \leq M \frac{e^{2\omega T} - 1}{2\omega} ||g||_{X_2}^2,
\]

where \( ||\cdot||_{X_2} \) denotes the norm in \( X_2 \). For any \( f \in Sp_{\sigma_2}(A) \),

\[
f = \sum_{k=1}^{m} E(\lambda_k, A) f
\]

and it holds that

\[
||f||^2 = (f, f) = \left( \sum_{k=1}^{m} E(\lambda_k, A) f, f \right) \leq \left( \sum_{k=1}^{m} ||E(\lambda_k, A) f||^2 \right)^{1/2} \left( \sum_{k=1}^{m} ||E^\dagger(\lambda_k, A) f||^2 \right)^{1/2}
\]

\[
\leq \left( \sum_{k=1}^{m} ||E(\lambda_k, A) f||^2 \right)^{1/2} \left( \sum_{k=1}^{m} ||E^\dagger(\lambda_k, A) f||^2 \right)^{1/2}
\]

\[
\leq \left( \sum_{k=1}^{m} ||E(\lambda_k, A) f||^2 \right)^{1/2} \left( M \frac{e^{2\omega T} - 1}{2\omega C_1} \right)^{1/2} ||f||.
\]

So

\[
||f||^2 \leq \frac{M \left( e^{2\omega T} - 1 \right)}{2\omega C_1} \sum_{k=1}^{m} ||E(\lambda_k, A) f||^2.
\]
Taking the limit, we have

\[ \|f\|^2 \leq \frac{M}{2\omega C_1} \left( e^{2\omega T} - 1 \right) \sum_{k=1}^{\infty} \|E(\lambda_k, A) f\|^2 \quad \forall f \in X_2. \]

Combining it with (2.10) yields

\[ \frac{2\omega C_1}{M(e^{2\omega T} - 1)} \sum_{k=1}^{\infty} \|E(\lambda_k, A) f\|^2 \leq \|f\|^2 \]

\[ \leq \frac{M(e^{2\omega T} - 1)}{2\omega C_1} \sum_{k=1}^{\infty} \|E(\lambda_k, A) f\|^2 \quad \forall f \in X_2. \quad (2.12) \]

This means that \{E(\lambda_k, A)X_2\}_{k \geq 1} is a subspace Riesz basis in \(X_2\). Also, (2.12) implies that \(\sum_{k=1}^{\infty} E(\lambda_k, A) f\) converges unconditionally in \(X_2\) with value \(f\), and yields that

\[ \sup_{n \geq 1} \left\| \sum_{k=1}^{n} E(\lambda_k, A) \right\|_{X_2} < \infty. \quad (2.13) \]

**Step 2:** There is a \(T(t)\)-invariant subspace \(X_1\) such that \(X = X_1 \oplus X_2\). Define a set in \(X\) by

\[ X_1 = Q(A) = \{ g \in X \mid E(\lambda_k, A) g = 0 \quad \forall k \geq 1 \}. \]

Evidently, \(X_1\) is a \(T(t)\)-invariant closed subspace, and \(X_1 \cap X_2 = \{0\}\). In order to prove that \(X = X_1 \oplus X_2\), it remains to show that \(X_1 + X_2\) is dense in \(X\). For this, we turn to the adjoint \(A^*\) of \(A\). Since \(X\) is a Hilbert space, so \(T^*(t)\) is also a \(C_0\) semigroup and its generator is \(A^*\). Moreover,

\[ \sigma(A^*) = \{ \lambda \mid \lambda \in \sigma(A) \} = \sigma_1(A) \cup \sigma_2(A) \]

and each \(\lambda_k \in \sigma_2(A)\) is an isolated eigenvalue of \(A^*\) with

\[ E^*(\lambda_k, A) = E(\lambda_k, A^*). \]
Denote
\[ X_2^* := \text{span}\{E^* (\lambda_k, A) X, k \geq 1\}, \]
\[ Q (A^*) := \{g \in X \mid E^* (\lambda_k, A) g = 0 \quad \forall k \geq 1\}. \]

It is easy to see that \( X_2^* \) and \( Q (A^*) \) are \( T^* (t) \)-invariant closed subspace and \( Q (A^*) \cap X_2^* = \{0\} \). Furthermore,
\[ X = Q (A^*) + X_2 \quad \text{and} \quad X = Q (A) + X_2^* \quad \text{(orthogonal sum)}. \]

Now, for any \( h \in X \) such that \( h \perp X_1 + X_2 \), we have \( h \perp X_1 \) and \( h \perp X_2 \). Therefore, \( h \in Q (A^*) \cap X_2^* \), and hence \( h = 0 \). This shows that \( X = X_1 \oplus X_2 \). Obviously, 
\[ \sigma(A|_{X_2}) = \sigma_2 (A) \quad \text{and} \quad \sigma(A|_{X_1}) = \sigma_1 (A), \]
so the first assertion of Theorem 1.1 follows from Steps 1 and 2.

**Step 3:** \( D (A) \subset X_1 \oplus X_2 \) when \( \sup_{k \geq 1} \| E (\lambda_k, A) \|_X < \infty \).

Suppose that \( \sup_{k \geq 1} \| E (\lambda_k, A) \|_X = M_1 < \infty \). Without loss of generality, we assume that \( 0 \in \rho (A) \). Then, for each \( f \in X \), we have
\[ E (\lambda_k, A) A^{-1} f = \sum_{j=1}^{m_k} \frac{(A - \lambda_k I)^{j-1} E (\lambda_k, A) f}{\lambda_k^j} \]
and
\[ \| E (\lambda_k, A) A^{-1} f \|^2 \leq \sum_{j=1}^{m_k} \frac{\| (A - \lambda_k I)^{j-1} E (\lambda_k, A) f \|}{(j - 1)!} \left( \sum_{j=1}^{m_k} \frac{(j - 1)!}{\lambda_k^j} \right)^2. \]

Eq. (2.9) implies that
\[ C_1 \sum_{j=0}^{m_k-1} \left( \frac{\| (A - \lambda_k I)^j E (\lambda_k, A) f \|}{j!} \right)^2 \leq M \frac{e^{2\omega T} - 1}{2\omega} ||E (\lambda_k, A) f||^2. \]

Eq. (2.4) implies that \( \sup_{k \geq 1} m_k = N < \infty \), and hence
\[ \sum_{j=1}^{m_k} \frac{(j - 1)!}{\lambda_k^j} \leq 2 \frac{(N!)^2}{|\lambda_k|^2}, \quad |\lambda_k| > 2. \]
 Altogether, we have
\[
\|E(\lambda_k, A)A^{-1}f\|^2 \leq \frac{M(e^{2\omega T} - 1)}{2\omega C_1} \frac{2(N!)^2}{|\lambda_k|^2} \|E(\lambda_k, A)f\|^2 \\
\leq \frac{M(e^{2\omega T} - 1)}{2\omega C_1} \frac{2(N!)^2}{|\lambda_k|^2} M_1^2 \|f\|^2.
\]

Since condition (2.6) together with \(\sup_{k} |\text{Re} \lambda_k| < \infty\) ensure that the series \(\sum_{k=1}^{\infty} |\lambda_k|^{-2}\) converges, so the series \(\sum_{k=1}^{\infty} \|E(\lambda_k, A)A^{-1}f\|^2\) also converges. From the first assertion of Theorem 1.1, we know that \(\sum_{k=1}^{\infty} E(\lambda_k, A)A^{-1}f\) is an element in \(X_2\). Setting
\[
g := A^{-1}f - \sum_{k=1}^{\infty} E(\lambda_k, A)A^{-1}f,
\]
we have \(g \in Q(A)\). So \(D(A) \subset X_1 \oplus X_2\) and the second assertion follows.

**Step 4:** The third assertion of Theorem 1.1 is true.

Suppose that \(X\) has a decomposition
\[
X = X_1 \oplus X_2.
\]

Define \(P\) be the projection from \(X\) to \(X_2\) along \(X_1\). Then \(P\) is a bounded operator on \(X\). For each \(f \in X\), we have \(Pf \in X_2\), and
\[
\left\| \sum_{k=1}^{n} E(\lambda_k, A)Pf \right\| \leq \left\| \sum_{k=1}^{n} E(\lambda_k, A) \right\|_{X_2} \| Pf \| \leq \left\| \sum_{k=1}^{n} E(\lambda_k, A) \right\|_{X_2} \| P \| \| f \|.
\]

From (2.13), we have
\[
\sup_{n \geq 1} \left\| \sum_{k=1}^{n} E(\lambda_k, A) \right\|_{X} < \infty.
\] (2.14)
Conversely, if (2.14) holds, then (2.10) and (2.14) imply that for any \( n \in \mathbb{N}, \ f \in X, \)
\[
C_1 \sum_{k=1}^{n} \| E(\lambda_k, A) f \|^2 \leq \frac{M(e^{2\omega T} - 1)}{2\omega} \left\| \sum_{k=1}^{n} E(\lambda_k, A) f \right\|^2 \\
\leq \frac{M(e^{2\omega T} - 1)}{2\omega} \left\| \sum_{k=1}^{n} E(\lambda_k, A) \right\|_X^2 \| f \|^2.
\]
Therefore,
\[
\sum_{k=1}^{\infty} \| E(\lambda_k, A) f \|^2 < \infty.
\]
This shows that for any \( f \in X, \) the series \( \sum_{k=1}^{\infty} E(\lambda_k, A) f \) converges unconditionally in \( X_2. \) If we define
\[
f_1 = f - \sum_{k=1}^{\infty} E(\lambda_k, A) f,
\]
then \( f_1 \in Q(A) = X_1. \) Thus, \( X = X_1 \oplus X_2 \) and the proof is then complete. \( \square \)

**Remark 2.1.** (1) In the proof of Theorem 1.1, we handle the Riesz projection sequence \( \{ E(\lambda_k, A) \}_{k \geq 1} \) very carefully. This is because we cannot obtain any information about it from the hypotheses. Although we can extend the family \( \{ E^{(n)}(n), n \geq 1 \} \) into \( E_1 \) such that \( E_1 \) forms a Riesz basis in \( L^2[0, T] \) (see Lemma 2.1) and obtain
\[
C_1 \sum_{k=1}^{\infty} \sum_{j=0}^{m_k} \left| \langle (T(t) f, g), h_{k,j} \rangle_{L^2[0,T]} \right|^2 \leq \frac{M(e^{2\omega T} - 1)}{2\omega} \| f \|^2 \| g \|^2
\]
with \( \{ h_{k,j}, 0 \leq j \leq m_k - 1 \} \) being the biorthogonal system associated with \( E^{(k)} \) and \( \langle \cdot, \cdot \rangle_{L^2[0,T]} \) being the inner product in \( L^2[0, T], \) we cannot derive from this that
\[
\langle (T(t) f, g), h_{k,0} \rangle_{L^2[0,T]} = (E(\lambda_k, A) f, g), \quad (2.15)
\]
which would imply the decomposition of \( X \) directly if it is true.

(2) Under some addition assumptions on the spectrum of \( A, \) for example, the conditions
\[
\sup \{ \text{Re} \lambda \mid \lambda \in \sigma_1(A) \} < \alpha < \beta < \inf \{ \text{Re} \lambda \mid \lambda \in \sigma_2(A) \}
\]
and

$$\sup_{x \leq \text{Re} \lambda \leq \beta} \| R(\lambda, A) \| < \infty,$$

there are some results about the decomposition of $X$ obtained in references such as [11,15]. In particular, [15] imposed an integral condition on the resolvents.

(3) In the proof of Theorem 1.1, we see that the dimension of the eigensubspace does not play a role. In fact, we only require that the chain length $m_k$ are uniformly bounded, so the assertions of Theorem 1.1 will still remain true if this condition holds.

**Definition 2.1.** An entire function $f(z)$ of exponential type $k$ is said to be of sine type if

1. the zeros of $f(z)$ are simple and separated;
2. there exist positive constants $A, B$ and $H$ such that

$$Ae^{k|y|} \leq |f(x + iy)| \leq Be^{k|y|}$$

whenever $x$ and $y$ are real and $|y| \geq H$.

According to this definition (cf. [30]), an entire function of sine type is bounded on the real axis and so must have infinitely many zeros. These zeros are all simple and lie in a strip parallel to the real axis. Furthermore, if $\{\lambda_n; n \geq 1\}$ is the zero set of $f(z)$, then there is a positive constant $\varepsilon > 0$ such that

$$|\lambda_n - \lambda_m| \geq \varepsilon, \quad n \neq m.$$

So condition (1.6) will always be true for entire functions of the sine type.

**Proof of Corollary 1.1.** Let $X$ be a separable Hilbert space, and let $A$ be an operator satisfying condition (1.7) and $\Gamma(z)$ be the associated entire function. Recall that condition (2) says that the spectrum of $A$ lies in the strip $|\text{Re} \lambda| \leq N$, and condition (3) reads that when $|z|$ is large enough, the zeros of $\Gamma(z)$ are simple and separated. Let $\{\lambda_k\}_{k=1}^\infty$ be an enumeration of all eigenvalues of $A$ with $|\text{Im} \lambda_k| \leq |\text{Im} \lambda_{k+1}|$. Then $\{\lambda_k\}_{k=1}^\infty$ satisfies the hypotheses of Theorem 1.1. Therefore, the family $\{E(\lambda_k, A)X_2\}_{k=1}^\infty$ forms a subspace Riesz basis for $X_2$. Note also that $\lambda_k$ is of chain length one as $k$ large enough, so we can always choose an orthonormal basis $\{\Phi_{k,j}; 1 \leq j \leq m_\sigma(\lambda_k)\}$ of the eigenspace of $A$ corresponding to each $\lambda_k$. Therefore these generalized eigenvectors form a Riesz basis for $\overline{Sp(A)}$. In particular, if the system of the generalized eigenvectors of $A$ is complete in $X$, then $X = \overline{Sp(A)}$ and the system $\{\Phi_{k,j}; 1 \leq j \leq m_\sigma(\lambda_k)\}_{k=1}^\infty$ forms a Riesz basis in $X$. The proof is then complete. \[\square\]
Remark 2.2. Note that without the completeness assumption of the generalized eigenvectors of \( A \), the assumptions on \( A \) and \( \Gamma(z) \) in Corollary 1.1 alone cannot guarantee that the generalized eigenvector is a Riesz basis for \( X \). For this, we refer to the literature [27].

3. Application to a heat exchanger equation with a boundary feedback

In this section, we shall present an application of our results on the following type of counter-flow heat exchanger equation with boundary feedback:

\[
\begin{align*}
\frac{\partial \theta_1}{\partial t} &= -v_1 \frac{\partial \theta_1}{\partial x} + h_1 (\theta_2 - \theta_1) \quad \text{for} \quad (t, x) \in (0, \infty) \times [0, \ell], \\
\frac{\partial \theta_2}{\partial t} &= v_2 \frac{\partial \theta_2}{\partial x} + h_2 (\theta_1 - \theta_2) \quad \text{for} \quad (t, x) \in (0, \infty) \times [0, \ell], \quad (3.1) \\
\theta_1(t, 0) &= -k_1 \theta_2(t, 0), \quad \theta_2(t, \ell) = -k_2 \theta_1(t, \ell) \quad \text{for} \quad t \in (0, \infty), \\
\theta_1(0, x) &= \theta_{10}(x), \quad \theta_2(0, x) = \theta_{20}(x) \quad \text{for} \quad x \in [0, \ell],
\end{align*}
\]

where \( v_1, v_2, h_1, h_2 \) and \( \ell \) are positive physical parameters, and \( k_1 \) and \( k_2 \) are feedback gains (in [16] they are assumed to be non-negative). For the sake of generality, we assume that \( k_1, k_2 \in \mathbb{R} \) and \( k_1 k_2 \neq 0 \).

Let \( \mathcal{H} := L^2[0, \ell] \times L^2[0, \ell] \) with inner product

\[
(f, g) := \int_0^\ell [f_1(x)g_1(x) + f_2(x)g_2(x)] \, dx, \quad f := [f_1, f_2]^T, \quad g := [g_1, g_2]^T \in \mathcal{H}.
\]

In \( \mathcal{H} \), operator \( A \) is defined by

\[
A f := \begin{bmatrix} -v_1 f_1' \\ v_2 f_2' \end{bmatrix} + \begin{bmatrix} -h_1 & h_1 \\ h_2 & -h_2 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \quad \text{for} \quad f \in \mathcal{D}(A), \quad (3.2)
\]

\[
\mathcal{D}(A) := \{ f = [f_1, f_2]^T \in \mathcal{H} | f_1', f_2' \in L^2[0, \ell], f_1(0) = -k_1 f_2(0), f_2(\ell) = -k_2 f_1(\ell) \}. \quad (3.3)
\]

Then, (3.1) can be written into an evolutionary equation in \( \mathcal{H} \)

\[
\begin{align*}
\frac{d}{dt} \Theta(t) &= A \Theta(t), \quad t > 0, \\
\Theta(0) &= \Theta_0
\end{align*} \quad (3.4)
\]
with \( \Theta(t) := [\theta_1(t, x), \theta_2(t, x)]^T \). Set

\[
\alpha_1 = \frac{h_1}{v_1}, \quad \beta_1 = \frac{h_2}{v_2}, \quad \alpha_2 = \frac{1}{v_1}, \quad \beta_2 = \frac{1}{v_2}.
\]

It has been proved in [16] that if the feedback gains \( k_1 \) and \( k_2 \) satisfy

\[
k_2^2 \leq \frac{\alpha_1}{\beta_1}, \quad k_2^2 \leq \frac{\beta_1}{\alpha_1},
\]

then the operator \( \mathcal{A} \) generates a uniformly bounded \( C_0 \) semigroup \( e^{\mathcal{A}t} \). In particular, for \( k_1 k_2 = 1 \), the exponential stability is obtained in [16]. However in the case that \( k_1 > 0, k_2 > 0 \) and \( k_1 k_2 \neq 1 \), due to the difficulty in calculating the spectrum, the stability of the system is unsolved. Here, we will use different approach to investigate the property of \( \mathcal{A} \) and to show that the system (3.4) is a Riesz system. For this purpose, we introduce an auxiliary operator \( \mathcal{A}_0 \) defined by

\[
\mathcal{A}_0 f := \begin{bmatrix} -v_1 f_1' \\ v_2 f_2' \end{bmatrix} + \begin{bmatrix} -h_1 & 0 \\ 0 & -h_2 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}
\]

with domain \( D(\mathcal{A}_0) = D(\mathcal{A}) \), and set \( \mathcal{B} := \mathcal{A} - \mathcal{A}_0 \). It is easy to see that \( \mathcal{B} \) is a bounded linear operator on \( \mathcal{H} \).

To prove that the eigenvector system of \( \mathcal{A}_0 \) forms a Riesz basis for \( \mathcal{H} \), we need a lemma.

**Lemma 3.1.** Let \( \gamma \in (0, 1) \) and \( E_n(x) := e^{2\pi \gamma i x} \). Then \( E_n(x) \) forms a Riesz basis for \( L^2[0, 1] \).

This lemma is a direct consequence of the fact that \( \{e^{2\pi \gamma i x}; n \in \mathbb{Z}\} \) is an orthonormal basis in \( L^2[0, 1] \).

**Theorem 3.1.** Let \( \mathcal{A}_0 \) be given in (3.6), then the following assertions are true.

1. \( \mathcal{A}_0 \) has a compact resolvent.
2. The spectrum of \( \mathcal{A}_0 \) is given by

\[
\lambda_n = \begin{cases} 
\frac{\ln(k_1 k_2) - (\alpha_1 + \beta_1)\ell}{(\alpha_2 + \beta_2)\ell} + \frac{2n\pi}{(\alpha_2 + \beta_2)\ell} i, & k_1 k_2 > 0, \quad n \in \mathbb{Z}, \\
\frac{\ln(k_1 k_2) - (\alpha_1 + \beta_1)\ell}{(\alpha_2 + \beta_2)\ell} + \frac{2(n+1)\pi}{(\alpha_2 + \beta_2)\ell} i, & k_1 k_2 < 0, \quad n \in \mathbb{Z},
\end{cases}
\]

where \( \mathbb{Z} \) is the integer set.
3. Each eigenvalue of \( \mathcal{A}_0 \) is simple, and the corresponding eigenvector system forms a Riesz basis for \( \mathcal{H} \).
Proof. It is easy to see that the operator $A_0$ has a compact resolvent. We calculate the eigenvalues and eigenvectors of $A_0$. Let $\lambda \in \mathbb{C}$ such that

$$A_0 f = \lambda f \quad \text{for } f \neq 0,$$

i.e.,

$$\begin{align*}
\lambda f_1(x) &= -v_1 f_1'(x) - h_1 f_1(x) \quad \forall x \in (0, \ell), \\
\lambda f_2(x) &= v_2 f_2'(x) - h_2 f_2(x) \quad \forall x \in (0, \ell), \\
f_1(0) &= -k_1 f_2(0) \quad f_2(\ell) = -k_2 f_1(\ell).
\end{align*}$$

Solving the equations, we obtain

$$\begin{align*}
f_1(x) &= -k_1 e^{-\left(\frac{\alpha_2 + \alpha_1}{\beta_2} \right) x}, \\
f_2(x) &= e^{\left(\frac{\alpha_2 + \beta_1}{\beta_2} \right) x}
\end{align*}$$

with $\lambda$ satisfying

$$e^{\left(\frac{\alpha_2 + \beta_1}{\beta_2} \right) \ell} - k_2 k_1 e^{-\left(\frac{\alpha_2 + \alpha_1}{\beta_2} \right) \ell} = 0.$$

Thus,

$$\lambda_n = \begin{cases}
\frac{\ln(k_1 k_2) - (\alpha_1 + \beta_1) \ell}{(\alpha_2 + \beta_2) \ell} + \frac{2n \pi}{(\alpha_2 + \beta_2) \ell} i, & k_1 k_2 > 0, \ n \in \mathbb{Z}, \\
\frac{\ln(k_1 k_2) - (\alpha_1 + \beta_1) \ell}{(\alpha_2 + \beta_2) \ell} + \frac{2(n+1) \pi}{(\alpha_2 + \beta_2) \ell} i, & k_1 k_2 < 0, \ n \in \mathbb{Z}.
\end{cases}$$

Obviously, each $\lambda_n$ is a simple and the corresponding eigenvector is

$$F_n = \begin{bmatrix}
-k_1 e^{-\left(\frac{\alpha_2 \lambda_n + \alpha_1}{\beta_2} \right) x} \\
e^{\left(\frac{\alpha_2 + \beta_1}{\beta_2} \right) x}
\end{bmatrix}, \quad n \in \mathbb{Z}.$$

We now show that $\{F_n; n \in \mathbb{Z}\}$ forms a Riesz basis for $\mathcal{H}$. Since

$$F_n = \begin{bmatrix}
-k_1 e^{\ln(k_1 k_2) - (\alpha_1 + \beta_1) \ell} x \\
0
\end{bmatrix}, \quad \begin{bmatrix}
\frac{2n \pi}{(\alpha_2 + \beta_2) \ell} x \\
\theta_2 e^{(\alpha_2 + \beta_2) \ell} x
\end{bmatrix}$$
and the operator $T$ defined, for $F = [f_1, f_2]^T \in \mathcal{H}$, by

$$
TF := \begin{bmatrix}
-k_1 e^{-\frac{\ln(k_1k_2) + (\alpha_2 - \alpha_1) \ell}{(\alpha_2 + \beta_2) \ell}} x & 0 \\
0 & e^{\frac{\ln(k_1k_2) - (\alpha_2 - \alpha_1) \ell}{(\alpha_2 + \beta_2) \ell}} x
\end{bmatrix} \begin{bmatrix}
f_1 \\
f_2
\end{bmatrix}
$$

is an invertible bounded linear operator, so we only need to prove that

$$
G_n = \begin{bmatrix}
e^{\frac{2n\alpha_2}{\alpha_2 + \beta_2} x^i} \\
e^{\frac{2n\beta_2}{\alpha_2 + \beta_2} x^i}
\end{bmatrix}, \quad n \in \mathbb{Z}
$$

forms a Riesz basis for $\mathcal{H}$. According to Lemma 3.1, $\{e^{\frac{2n\alpha_2}{\alpha_2 + \beta_2} x^i} \mid n \in \mathbb{Z}\}$ and $\{e^{\frac{2n\beta_2}{\alpha_2 + \beta_2} x^i} \mid n \in \mathbb{Z}\}$ are Riesz bases for $L^2[0, \ell]$, so $\{G_n; n \in \mathbb{Z}\}$ also forms a Riesz basis for $\mathcal{H}$, and the proof is then completed. □

**Corollary 3.1.** Let $A_0$ be defined by (3.6) and $k_1k_1 \neq 0$. Then $A_0$ generates a $C_0$ group in $\mathcal{H}$, and so $A$ also generates a $C_0$ group.

**Proof.** According to Theorem 3.1, $\{F_n; n \in \mathbb{Z}\}$ forms a Riesz basis in $\mathcal{H}$. Let $\{F^*_n; n \in \mathbb{Z}\}$ be the biorthogonal system associated with $\{F_n; n \in \mathbb{Z}\}$. We define an operator $S(t)$ by

$$
S(t)f := \sum_{n\in\mathbb{Z}} e^{i\alpha_n t} (f, F^*_n) F_n \quad \forall f = [f_1, f_2]^T \in \mathcal{H}.
$$

(3.8)

Since

$$
|e^{i\alpha_n t}| \leq \exp \left\{ t \left[ \frac{|\ln(k_1k_2)| + (\alpha_1 + \beta_1) \ell}{(\alpha_2 + \beta_2) \ell} \right] \right\} \quad \forall n \in \mathbb{Z},
$$

so $S(t)f$ in (3.8) is well defined. Clearly, $S(t)$ is a strongly continuous group. So the perturbation theory of semigroup (e.g., see [21]) says that $A = A_0 + B$ is a generator of a $C_0$ group. □
In order to study the property of \( \mathcal{A} \), we need the following Lemma.

**Lemma 3.2** (Xu and Wang [28]). Let \( X \) be a separable Hilbert space and \( \mathcal{A} \) be the generator of a \( C_0 \) semigroup. Suppose that \( R(\lambda, \mathcal{A}) \) is a Riesz operator and there exist a real number \( \rho \) and \([\rho]+1\) rays \( \mu_0, \mu_1, \mu_2, \ldots, \mu_{[\rho]} \) such that

\[
\|R(\lambda, \mathcal{A})\| = O(e^{\|\lambda\|^\rho}),
\]

\[
\arg \mu_0 = \frac{\pi}{2} < \arg \mu_1 < \arg \mu_2 < \cdots < \arg \mu_{[\rho]} = \frac{3\pi}{2}
\]

and

\[
\arg \mu_{k+1} - \arg \mu_k < \frac{\pi}{2([\rho]+1)}.
\]

If \( R(\lambda, \mathcal{A}) \) is bounded on each \( \mu_j, j = 1, 2, 3, \ldots, [\rho]-1 \), then the system of generalized eigenvectors of \( \mathcal{A} \) is complete in \( X \).

**Corollary 3.2.** Let \( \mathcal{A} \) be defined by (3.2) and (3.3). Then the following assertions are true:

1. \( \mathcal{A} \) has a compact resolvent.
2. If \( k_1k_2 \neq 0 \), then the family of generalized eigenvectors of \( \mathcal{A} \) is complete in \( \mathcal{H} \).

**Proof.** Let \( \lambda \in \mathbb{C} \). For any \( g := [g_1, g_2]^T \in \mathcal{H} \), we consider the resolvent problem:

\[
(\lambda I - \mathcal{A})f = g,
\]

i.e.,

\[
\begin{align*}
\lambda f_1 + v_1 f_1' + h_1 f_1 - h_1 f_2 &= g_1, \\
\lambda f_2 - v_2 f_2' - h_2 f_1 + h_2 f_2 &= g_2
\end{align*}
\]

with boundary conditions \( f_1(0) + k_1 f_2(0) = 0, f_2(\ell) + k_2 f_1(\ell) = 0 \). Under the previous notations, we have

\[
\frac{d}{dx} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} + \begin{bmatrix} \lambda x_2 + x_1 & -x_1 \\ \beta_1 & -(\lambda \beta_2 + \beta_1) \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} x_2 g_1 \\ -\beta_2 g_2 \end{bmatrix}.
\]

Solving them yields

\[
\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \exp \{-A(\lambda)x\} \begin{bmatrix} f_1(0) \\ f_2(0) \end{bmatrix} + \int_0^x \exp \{-A(\lambda)(x-s)\} \begin{bmatrix} x_2 g_1(s) \\ -\beta_2 g_2(s) \end{bmatrix} ds.
\]
where

\[ A(\lambda) = \begin{bmatrix} \lambda \alpha_2 + \alpha_1 & -\alpha_1 \\ \beta_1 & -(\lambda \beta_2 + \beta_1) \end{bmatrix}. \]

Using the boundary conditions, we obtain

\[ f_2(0)[k_2 1] \exp[-A(\lambda)\ell] \begin{bmatrix} 1 \\ -k_1 \end{bmatrix} + \int_0^\ell [k_2 1] \exp[-A(\lambda)(x-s)] \begin{bmatrix} \alpha_2 g_1(s) \\ -\beta_2 g_2(s) \end{bmatrix} ds = 0. \]

If \( \Gamma(\lambda) = [k_2 1] \exp[-A(\lambda)\ell] \begin{bmatrix} 1 \\ -k_1 \end{bmatrix} \neq 0 \), then the solution can be expressed into

\[
\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = -\frac{1}{\Gamma(\lambda)} \int_0^\ell \exp[-A(\lambda)x] \begin{bmatrix} k_2 & 1 \\ -k_1 k_2 & -k_1 \end{bmatrix} \exp[-A(\lambda)(\ell-s)] \begin{bmatrix} \alpha_2 g_1(s) \\ -\beta_2 g_2(s) \end{bmatrix} ds \\
+ \int_0^x \exp[-A(\lambda)(x-s)] \begin{bmatrix} \alpha_2 g_1(s) \\ -\beta_2 g_2(s) \end{bmatrix} ds = R(\lambda, A) g.
\]

So, we deduce that \( R(\lambda, A) \) is compact on \( \mathcal{H} \) and

\[ ||R(\lambda, A)|| = O(e^{2\ell (\alpha_2 + \beta_2) |\lambda|}). \]

From Corollary 3.1, \( R(\lambda, A) \) is bounded on \( \arg \lambda = \pi \) as \( k_1 k_2 \neq 0 \). Therefore, Lemma 3.2 yields the completeness of the generalized eigenvectors of \( A \). \( \square \)

To investigate the eigenvalue problem of \( A \), let \( \lambda \in \sigma(A) \), \( f = [f_1, f_2]^T \in \mathcal{H} \) such that \( (\lambda I - A) f = 0 \), i.e.,

\[
\begin{align*}
\dot{f}_1 + v_1 f_1' + h_1 f_1 - h_1 f_2 &= 0, \\
\dot{f}_2 - v_2 f_2' - h_2 f_1 + h_2 f_2 &= 0, \\
f_1(0) + k_1 f_2(0) &= 0, \quad f_2(\ell) + k_2 f_1(\ell) = 0.
\end{align*}
\]

\[ (3.9) \]
Then, $f_1$ and $f_2$ are the solution of the equation:

$$y'' + [(x_2 \lambda + x_1) - (\beta_2 \lambda + \beta_1)]y' + [x_1 \beta_1 - (x_1 \lambda + x_1)(\beta_2 \lambda + \beta_1)]y = 0.$$ 

Denote

$$b(\lambda) := \frac{1}{2}[(x_2 \lambda + x_1) - (\beta_2 \lambda + \beta_1)],$$

(3.10)

$$w(\lambda) := \frac{1}{2} \sqrt{[x_1 + \beta_1 + (x_2 + \beta_2)\lambda]^2 - 4x_1 \beta_1},$$

(3.11)

$$v(\lambda) := \frac{(x_1 + \beta_1 + (x_2 + \beta_2)\lambda)}{2}.$$  

(3.12)

Set

$$f_j(x) := \tilde{c}_j e^{(-b(\lambda) + w(\lambda))x} + \eta_j e^{(-b(\lambda) - w(\lambda))x}, \quad j = 1, 2.$$ 

Substituting $f_j(x)$ into (3.9) leads to some algebraic equations:

$$\begin{align*}
[v(\lambda) + w(\lambda)]\xi_1 - x_1 \xi_2 &= 0, \\
[v(\lambda) - w(\lambda)]\eta_1 - x_1 \eta_2 &= 0, \\
\xi_1 + \eta_1 + k_1(\xi_2 + \eta_2) &= 0, \\
(k_2 \xi_1 + \xi_2)e^{(-b(\lambda) + w(\lambda))\ell} + (k_2 \eta_1 + \eta_2)e^{(-b(\lambda) - w(\lambda))\ell} &= 0.
\end{align*}$$

So, equations in (3.9) will have non-zero solutions if and only if

$$\Gamma(\lambda):= \det \begin{bmatrix} x_1 + k_1[v(\lambda) + w(\lambda)] & x_1 + k_1[v(\lambda) - w(\lambda)] \\
x_1 k_2 + v(\lambda) + w(\lambda) & x_1 k_2 + v(\lambda) - w(\lambda) \end{bmatrix} e^{w(\lambda)\ell} = 0,$$

i.e.,

$$0 = \Gamma(\lambda) = [x_1 + k_1(v(\lambda) + w(\lambda))] [x_1 k_2 + v(\lambda) - w(\lambda)] e^{-w(\lambda)\ell}$$

$$- [x_1 + k_1(v(\lambda) - w(\lambda))] [x_1 k_2 + v(\lambda) + w(\lambda)] e^{w(\lambda)\ell}. \quad (3.13)$$

From (3.11) and (3.12), we have $w(\lambda) - v(\lambda) = O(\lambda^{-1})$ when $|\lambda|$ is sufficiently large. So

$$\Gamma(\lambda) = [x_1 k_1 k_2(x_2 + \beta_2)\lambda + O(1)]e^{-v(\lambda)\ell} - [x_1 (x_2 + \beta_2)\lambda + O(1)]e^{v(\lambda)\ell}.$$
Set

\[ G(\lambda) := k_1 k_2 e^{-\nu(\lambda)t} - e^{\nu(\lambda)t}, \]

then we have

\[ G(\lambda) - \frac{\Gamma(\lambda)}{\alpha(\alpha^2 + \beta^2)} = O(\lambda^{-1}) \quad (3.14) \]

as \(|\lambda|\) large enough. With these, we are all set to prove the following result.

**Theorem 3.2.** Let \(A\) be defined as before and \(k_1 k_2 \neq 0\). Then the following assertions hold.

1. For each \(\lambda \in \sigma(A)\), \(\dim N(\lambda I - A) = 1\).
2. For \(\lambda \in \sigma(A)\), the chain length \(m(\lambda)\) is 1 when \(|\lambda|\) is large enough, and

\[ \lambda = \lambda_n + O(\lambda_n^{-1}) \quad (3.15) \]

with \(\lambda_n\) given in (3.7).
3. System (3.4) is a Riesz system.

**Proof.** From the previous discussion, we know that the first assertion is true. Also, we note that each \(\lambda_n, n \in \mathbb{Z}\), defined by (3.7) is a simple zero of \(G(\lambda)\). Applying Rouché’s Theorem to (3.14), we conclude that \(\Gamma(\lambda)\) has simple zero \(\xi_n\) very close to \(\lambda_n\) as \(|\lambda|\) is large enough. Set

\[ \xi_n := \lambda_n + \eta_n \]

and substitute \(\xi_n\) into (3.14) yields \(G(\xi_n) = O(\xi_n^{-1})\). Expanding \(G(\lambda)\) at \(\lambda_n\), we have

\[ G(\xi_n) = G'(\lambda_n)(\xi_n - \lambda_n) + O((\xi_n - \lambda_n)^2) = O(\xi_n^{-1}). \]

So the estimate (3.15) follows. Since (3.15) implies the separability of spectrum, so the hypotheses of Theorem 1.1 are fulfilled. Combining Theorem 1.1 with the completeness in Corollary 3.2, we conclude that (3.4) is a Riesz system. \(\square\)

**Acknowledgments**

The authors thank the referees for their very useful and helpful comments and suggestions.
References

[24] M.A. Shubov, The Riesz basis property of the system of root vectors for the equation of a
nonhomogeneous damped string: transformation operators method, Methods Appl. Anal. 6 (4) (1999)
571–591.
[25] M.A. Shubov, Asymptotic and spectral analysis of the spatially nonhomogeneous Timoshenko beam
[28] G.Q. Xu, S.H. Wang, The completeness of systems of generalized eigenfunctions of generators of
[29] G.Q. Xu, D.X. Feng, The Riesz basis property of a Timoshenko beam with boundary feedback and