1. **Basic Control Signals**

There are many signals used for testing the control systems. Some of them are called the basic signals, such as the step function, impulse function, ramp function and sinusoidal function. These signals are of major importance for control applications.

1.1 **Unit Step Function**

The unit step function is designated by \( u(t - T) \) and is defined as follows:

\[
\begin{align*}
 u(t - T) &= \begin{cases} 
 1 & \text{for } t > T \\
 0 & \text{for } t < T \\
 \text{undefined} & \text{for } t = T 
\end{cases}
\]

The graphical representation of the unit step function is shown in Fig. 2.1. The amplitude of \( u(t - T) \), for \( t > T \), is equal to 1. This is why the function \( u(t - T) \) is called the “unit” step function. If the value of the function equals something rather than "1", it called Step function.
Fig. 2.1 Unit step function with delay

A physical example of the step function is the electrical switch of the circuit shown in Fig. 2.2. It is obvious that the voltage \( v_R(t) \) is given by:

\[
v_R(t) = \begin{cases} 
  v(t) & \text{for } t > T \\
  0 & \text{for } t < T \\
  \text{undefined} & \text{for } t = T
\end{cases}
\]

OR

\[
v_R(t) = v(t)u(t - T)
\]

Fig. 2.2 Electrical switch represent a step function

The unit gate function \( g(t) \), shown in Fig. 2.3, can be derived from the unit step function as \( g(t) = u(t - T_1) - u(t - T_2) \) where \( T_1 < T_2 \), and is defined as follows:

\[
g_x(t) = \begin{cases} 
  1 & \text{for } t \in (T_1, T_2) \\
  0 & \text{for } t \notin (T_1, T_2) \\
  \text{undefined} & \text{for } t = T_1 \text{ and } t = T_2
\end{cases}
\]

Fig. 2.3 Unit gate function

The unit gate function is usually used to zero all values of another function, outside a certain time interval. Consider for example the function \( f(t) \), then, the function \( y(t) = f(t) \times g(t) \) is represented as follows:
1.2 Unit Impulse Function

The unit impulse function, which is also called the Dirac function, is designated by $\delta(t - T)$ and is defined as follows:

$$\delta(t - T) = \begin{cases} 0 & \forall t, \text{ except for } t = T \\ \infty & \text{for } t = T \end{cases}$$

The graphical representation of $\delta(t - T)$ is given in Fig. 2.4. In Fig. 2.5 $\delta(t - T)$ is defined in a different way as follows: the area $c(t)$ of the parallelogram is

$$c(t) = (1/a)a = 1.$$  

As $a$ becomes larger, the base of the parallelogram $1/a$ becomes smaller. In the limit, as the height $a$ tends to infinity, the base $1/a$ tends to zero, i.e.,

$$\delta(t - T) \text{ occurs when } \lim_{a \to \infty} c(t)$$

From the above definition, we conclude that

$$\int_{-\infty}^{\infty} \delta(t - T) \, dt = 1 = u(t - T)$$

This shows that the area of the unit impulse function is equal to 1(this is why it is called the “unit” impulse function).

The functions $u(t - T)$ and $\delta(t - T)$ are related as follows:

$$\delta(t - T) = \frac{du(t - T)}{dt} \quad \text{and} \quad u(t - T) = \int_{-\infty}^{t} \delta(\lambda - T) \, d\lambda.$$

Fig. 2.4 Unit impulse function

Fig. 2.5 Area of function $c(t)$ used in explanation of impulse function
1.3 The Ramp Function

The ramp function is designated by \( r(t - T) \) and is defined as follows:

\[
 r(t - T) = \begin{cases} 
 t - T & \text{for } t > T \\
 0 & \text{for } t \leq T 
\end{cases}
\]

The graphical representation of \( r(t - T) \) is shown in Fig. 2.6. It is obvious that \( u(t - T) \) and \( r(t - T) \) are related as follows:

\[
 u(t - T) = \frac{dr(t - T)}{dt} \quad \text{and} \quad r(t - T) = \int_{-\infty}^{t} u(\lambda - T) \, d\lambda
\]

![Fig. 2.6 Unit ramp function](image)

**Note:** All the above functions are usually applied when \( T = 0 \). In cases of \( T > 0 \), then the function is delayed by \( T \) units of time. Whereas when \( T < 0 \), the function is preceding by \( T \) units of time.

2. Laplace Transform

To study and design any control system, one relies to a great extent on a set of mathematical tools. As example of these mathematical tools is the Laplace transform which is very important for the study and design of such systems. The definition of the Laplace transform of a function \( f(t) \) is as follows:

\[
 L\{f(t)\} = \int_{0}^{\infty} f(t)e^{-st} \, dt = F(s)
\]

where \( L \) designates the Laplace transform and \( s \) is the complex variable defined as \( \sigma + j\omega \). Usually, the time function \( f(t) \) is written with a small \( f \), while the complex variable function \( F(s) \) is written with a capital \( F \).
Let \( L\{ f(t) \} = F(s) \). Then, the inverse Laplace transform of \( F(s) \) is also a linear integral transform, defined as follows:

\[
L^{-1}\{ F(s) \} = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} F(s)e^{st} \, ds = f(t)
\]

where \( L^{-1} \) designates the inverse Laplace transform, \( j = \sqrt{-1} \), and \( c \) is a complex constant.

Clearly, the Laplace transform is a mathematical tool which transforms a function from one domain to another. In particular, it transforms a time-domain function to a function in the frequency domain and vice versa. This gives the flexibility to study a function in both the time domain and the frequency domain, which results in a better understanding of the function, its properties, and its time-domain, frequency-domain properties.

For example, consider the exponential function shown

\[
f(t) = \begin{cases} 
0 & \text{for } t < 0 \\
Ae^{-\alpha t} & \text{for } t \geq 0
\end{cases}
\]

where \( A \) and \( \alpha \) are constants. The Laplace transform of \( f(t) \) can be obtained as follows.
Another example, consider the Step function with value A as shown

\[ f(t) = \begin{cases} 
0 & \text{for } t < 0 \\
A & \text{for } t > 0
\end{cases} \]

\[ F(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{A\} = \int_0^\infty A e^{-st} \, dt = \frac{A}{s} \]

For the unit step function, substitute A = 1 in the above equation, F(s) = 1/S

Another example, consider the Ramp function with value A as shown

\[ f(t) = \begin{cases} 
0 & \text{for } t < 0 \\
At & \text{for } t \geq 0
\end{cases} \]

\[ F(s) = \mathcal{L}\{At\} = A \int_0^\infty te^{-st} \, dt \]

To evaluate the above integral we use the formula for the integration by parts

\[ \int_0^\infty udv = uv \bigg|_0^\infty - \int_0^\infty v \, du \]

\[ F(s) = \mathcal{L}\{At\} = \int_0^\infty Ate^{-st} \, dt = \frac{Ate^{-st}}{s} \bigg|_0^\infty - \frac{Ae^{-st}}{s} \bigg|_0^\infty = \frac{A}{s^2} \]

For unit ramp, substitute A = 1 in the above equation. F(s) = 1/S^2
2.1 Complex Variable S

A complex variable $S$ has two components: a real component $\sigma$ and an imaginary component $j\omega$. Graphically, the real component of $s$ is represented by a $\sigma$ axis in the horizontal direction, and the imaginary component is measured along the vertical $j\omega$ axis, in the complex $s$-plane. Fig. 2.7 illustrates the complex $s$-plane, in which any arbitrary point $s = s_1$ is defined by the coordinates $\sigma = \sigma_1$, and $j\omega = j\omega_1$, or simply $s_1 = \sigma_1 + j\omega_1$.

![Fig. 2.7 The complex S-plan](image)

2.2 Pole-Zero Map

Consider a complex function $G(S)$. Therefore, $G(S)$ is said to be analytic in a region if $G(S)$ and all of its derivatives exist in that region.

$$ G(s) = \frac{1}{S + 1} $$

Points in the S plane at which the function $G(s)$ is analytic are called ordinary points, while points in the s plane at which the function $G(s)$ is not analytic are called singular points. Singular points at which the function $G(s)$ or its derivatives approach infinity are called poles. In the previous example, $S = -1$ is a singular point and is a pole of the function $G(s)$.

Points at which the function $G(s) = 0$ are called zeros.

Consider the function

$$ G(s) = \frac{K(S + 2)(S + 10)}{S(S + 1)(S + 5)(S + 15)^2} $$

$G(s)$ has zeros at $s = -2$, $s = -10$, also has poles at $s = 0$, $s = -1$, $s = -5$ and double poles at $s = -15$. Note that, $G(s)$ becomes zero at a large value of $s = \infty$.

If the points at infinity are included, $G(s)$ has the same number of poles as zeros.

five zeros ($s = -2$, $s = -10$, $S = \infty$, $S = \infty$, $S = \infty$) and five poles ($s = 0$, $s = -1$, $s = -5$, $s = -15$, and $s = -15$)
2.3 Properties and Theorems of the Laplace Transform

2.3.1 Linearity

Laplace transform is a linear transformation, the following relation holds

\[ L\{c_1 f_1(t) + c_2 f_2(t)\} = L\{c_1 f_1(t)\} + L\{c_2 f_2(t)\} = c_1 F_1(s) + c_2 F_2(s) \]

where \( c_1 \) and \( c_2 \) are constants, \( F_1(s) = L\{f_1(t)\} \) and \( F_2(s) = L\{f_2(t)\} \).

2.3.2 Laplace Transform of Derivative of a Function

Let \( f^{(1)}(t) \) be the time derivative of \( f(t) \), and \( F(s) \) be the Laplace transform of \( f(t) \). Then, the Laplace transform of \( f^{(1)}(t) \) is given by

\[ L\{f^{(1)}(t)\} = sF(s) - f(0) \]

Where \( f(0) \) is the initial value at \( (t = 0) \) of the function \( f(t) \).

Working in the same way for the Laplace transform of the second derivative

\[ L\{f^{(2)}(t)\} = s^2 F(s) - sf(0) - f^{(1)}(0) \]

For the general case we have

\[ L\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f^{(1)}(0) - \ldots - f^{(n-1)}(0) \]

2.3.3 Laplace Transform of Integral of a Function

Let \( \int_{\gamma}^{t} f(\lambda) \, d\lambda \) be the integral of a function \( f(t) \), where \( \gamma \) is a positive number and \( F(s) \) is the Laplace transform of \( f(t) \). Then, the Laplace transform of the integral is given by

\[ L\left\{ \int_{\gamma}^{t} f(\lambda) \, d\lambda \right\} = \frac{F(s)}{s} \]

For the \( n^{th} \) integral of the function \( f(t) \) we have,

\[ L\left\{ \int_{\gamma}^{t} \ldots \int_{\gamma}^{t} f(\lambda) (d\lambda)^n \middle|_{s \text{ times}} \right\} = \frac{F(s)}{s^n} \]

Therefore, assuming zero initial conditions of the function \( f(t) \), then
Consider the function \( f(t) \). Then, the function \( f(t-T) \) is the same function shifted to the right of \( f(t) \) by \( T \) units (Fig. 2.8). The Laplace transform of the initial function \( f(t) \) and of the shifted (delayed) function \( f(t-T) \), are related as follows:

\[
L\{f(t-T)u(t-T)\} = e^{-sT}F(s)
\]

The important feature of the Laplace transform is that it greatly simplifies the procedure of taking the derivative and/or the integral of a function \( f(t) \). Indeed, the Laplace transform “transforms” the derivative of \( f(t) \) in the time domain into multiplying \( F(s) \) by \( s \) in the frequency domain. Furthermore, it “transforms” the integral of \( f(t) \) in the time domain into dividing \( F(s) \) by \( s \) in the frequency domain.

### 2.3.4 Time Scaling

Consider the functions \( f(t) \) and \( f(at) \), where \( a \) is a positive number. The function \( f(at) \) differs from \( f(t) \), in time scaling, by \( a \) units. For these two functions, it holds:

\[
L\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right)
\]

### 2.3.5 Shift in the Frequency Domain

It holds that:

\[
L\{e^{-at}f(t)\} = F(s + a)
\]

This means, the Laplace transform of the product of the functions \( e^{-at} \) and \( f(t) \), leads to shifting of \( F(s) = L\{f(t)\} \) by \( a \) units.

### 2.3.6 Shift in the Time Domain

Consider the function \( f(t)u(t) \). Then, the function \( f(t-T)u(t-T) \) is the same function shifted to the right of \( f(t)u(t) \) by \( T \) units (Fig. 2.8). The Laplace transform of the initial function \( f(t)u(t) \) and of the shifted (delayed) function \( f(t-T)u(t-T) \), are related as follows:

\[
L\{f(t-T)u(t-T)\} = e^{-sT}V(s)
\]
2.3.7 The Initial Value Theorem
This theorem refers to the behavior of the function \( f(t) \) as \( t \to 0 \) and, for this reason, is called the initial value theorem. This theorem is given by the relation

\[
\lim_{t \to 0} f(t) = \lim_{s \to \infty} sF(s)
\]

2.3.8 The Final Value Theorem
This theorem refers to the behavior of the function \( f(t) \) as \( t \to \infty \) and, for this reason, it is called the final value theorem. This theorem is given by the relation

\[
\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)
\]

**Common partial fraction expansions**

(i) Factored roots

\[
\frac{K}{s(s + a)} = \frac{A}{s} + \frac{B}{s + a}
\]

(ii) Repeated roots

\[
\frac{K}{s^2(s + a)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s + a}
\]

(iii) Second-order real roots \((b^2 > 4ac)\)

\[
\frac{K}{s(as^2 + bs + c)} = \frac{K}{s(s + d)(s + e)} = \frac{A}{s} + \frac{B}{s + d} + \frac{C}{s + e}
\]

(iv) Second-order complex roots \((b^2 < 4ac)\)

\[
\frac{K}{s(as^2 + bs + c)} = \frac{A}{s} + \frac{Bs + C}{as^2 + bs + c}
\]

Completing the square gives

\[
\frac{A}{s} + \frac{Bs + C}{(s + \alpha)^2 + \omega^2}
\]
Examples

Consider the exponential function

\[ f(t) = 0, \quad \text{for } t < 0 \]
\[ = Ae^{-at}, \quad \text{for } t \geq 0 \]

where \( A \) and \( a \) are constants. The Laplace transform of this exponential function can be obtained as follows:

\[
\mathcal{L}[e^{-at}] = \int_0^\infty Ae^{-at}e^{-st} \, dt = A \int_0^\infty e^{-(a+s)t} \, dt = \frac{A}{s+a}
\]

Consider the step function

\[ f(t) = 0, \quad \text{for } t < 0 \]
\[ = A, \quad \text{for } t > 0 \]

where \( A \) is a constant. Note that it is a special case of the exponential function \( Ae^{-at} \), where \( a = 0 \). The step function is undefined at \( t = 0 \). Its Laplace transform is given by

\[
\mathcal{L}[A] = \int_0^\infty Ae^{-st} \, dt = \frac{A}{s}
\]

Consider the ramp function

\[ f(t) = 0, \quad \text{for } t < 0 \]
\[ = At, \quad \text{for } t \geq 0 \]

where \( A \) is a constant. The Laplace transform of this ramp function is obtained as

\[
\mathcal{L}[At] = \int_0^\infty Ate^{-st} \, dt = A \left[ \frac{e^{-st}}{-s} \right]_0^\infty - \int_0^\infty \frac{Ae^{-st}}{-s} \, dt
\]
\[ = \frac{A}{s} \int_0^\infty e^{-st} \, dt = \frac{A}{s^2}
\]

The Laplace transform of the sinusoidal function

\[ f(t) = 0, \quad \text{for } t < 0 \]
\[ = A \sin \omega t, \quad \text{for } t \geq 0 \]

where \( A \) and \( \omega \) are constants, is obtained as follows. Referring to Equation (2-3), \( \sin \omega t \)

can be written

\[
\sin \omega t = \frac{1}{2j} (e^{j\omega t} - e^{-j\omega t})
\]

Hence

\[
\mathcal{L}[A \sin \omega t] = \frac{A}{2j} \int_0^\infty (e^{j\omega t} - e^{-j\omega t})e^{-st} \, dt
\]
\[ = \frac{A}{2j} \left[ \frac{1}{s-j\omega} - \frac{1}{s+j\omega} \right] = \frac{A\omega}{s^2 + \omega^2}
\]

Similarly, the Laplace transform of \( A \cos \omega t \) can be derived as follows:

\[
\mathcal{L}[A \cos \omega t] = \frac{As}{s^2 + \omega^2}
\]
Known that
\[ \mathcal{L}[\sin \omega t] = \frac{\omega}{s^2 + \omega^2} = F(s), \quad \mathcal{L}[\cos \omega t] = \frac{s}{s^2 + \omega^2} = G(s) \]

Therefore, multiplying these functions by exponential function give that:
\[ \mathcal{L}[e^{-\alpha t} \sin \omega t] = F(s + \alpha) = \frac{\omega}{(s + \alpha)^2 + \omega^2} \]
\[ \mathcal{L}[e^{-\alpha t} \cos \omega t] = G(s + \alpha) = \frac{s + \alpha}{(s + \alpha)^2 + \omega^2} \]

Given that
\[ \mathcal{L}[f(t)] = F(s) = \frac{1}{s(s + 1)} \]

Based on final value theorem
\[ \lim_{t \to \infty} f(t) = f(\infty) = \lim_{s \to 0} sF(s) = \lim_{s \to 0} \frac{s}{s + 1} = \lim_{s \to 0} \frac{1}{s + 1} = 1 \]

Find the inverse Laplace transform of
\[ F(s) = \frac{s + 3}{(s + 1)(s + 2)} \]

The partial-fraction expansion of \( F(s) \) is
\[ F(s) = \frac{s + 3}{(s + 1)(s + 2)} = \frac{a_1}{s + 1} + \frac{a_2}{s + 2} \]

where \( a_1 \) and \( a_2 \) are found by using Equation (2–15):
\[ a_1 = \left[ \frac{s + 3}{(s + 1)(s + 2)} \right]_{s=-1} = \left[ \frac{s + 3}{s + 2} \right]_{s=-1} = 2 \]
\[ a_2 = \left[ \frac{s + 3}{(s + 1)(s + 2)} \right]_{s=-2} = \left[ \frac{s + 3}{s + 1} \right]_{s=-2} = -1 \]

Thus
\[ f(t) = \mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}\left[ \frac{2}{s + 1} \right] + \mathcal{L}^{-1}\left[ \frac{-1}{s + 2} \right] = 2e^{-t} - e^{-2t}, \quad \text{for } t \geq 0 \]
Obtain the inverse Laplace transform of

\[ G(s) = \frac{s^3 + 5s^2 + 9s + 7}{(s + 1)(s + 2)} \]

Here, since the degree of the numerator polynomial is higher than that of the denominator polynomial, we must divide the numerator by the denominator.

\[ G(s) = \frac{s}{s + 2} + \frac{s + 3}{(s + 1)(s + 2)} \]

Note that the Laplace transform of the unit-impulse function \( \delta(t) \) is 1 and that the Laplace transform of \( d\delta(t)/dt \) is \( s \). The third term on the right-hand side of this last equation is \( F(s) \) in Example 2–3. So the inverse Laplace transform of \( G(s) \) is given as

\[ g(t) = \frac{d}{dt} \delta(t) + 2\delta(t) + 2e^{-t} - e^{-2t}, \quad \text{for } t \geq 0 \]

Find the inverse Laplace transform of

\[ F(s) = \frac{2s + 12}{s^2 + 2s + 5} \]

Notice that the denominator polynomial can be factored as

\[ s^2 + 2s + 5 = (s + 1 + j2)(s + 1 - j2) \]

If the function \( F(s) \) involves a pair of complex-conjugate poles, it is convenient not to expand \( F(s) \) into the usual partial fractions but to expand it into the sum of a damped sine and a damped cosine function.

Noting that \( s^2 + 2s + 5 = (s + 1)^2 + 2^2 \) and referring to the Laplace transforms of \( e^{-\alpha t} \sin \omega t \) and \( e^{-\alpha t} \cos \omega t \), rewritten thus,

\[ \mathcal{L}[e^{-\alpha t} \sin \omega t] = \frac{\omega}{(s + \alpha)^2 + \omega^2} \]

\[ \mathcal{L}[e^{-\alpha t} \cos \omega t] = \frac{s + \alpha}{(s + \alpha)^2 + \omega^2} \]

the given \( F(s) \) can be written as a sum of a damped sine and a damped cosine function.

\[ F(s) = \frac{2s + 12}{s^2 + 2s + 5} = \frac{10 + 2(s + 1)}{(s + 1)^2 + 2^2} \]

\[ = 5 \left[ \frac{2}{(s + 1)^2 + 2^2} + \frac{s + 1}{(s + 1)^2 + 2^2} \right] \]

It follows that

\[ f(t) = \mathcal{L}^{-1}[F(s)] \]

\[ = 5\mathcal{L}^{-1}\left[ \frac{2}{(s + 1)^2 + 2^2} \right] + 2\mathcal{L}^{-1}\left[ \frac{s + 1}{(s + 1)^2 + 2^2} \right] \]

\[ = 5e^{-t}\sin 2t + 2e^{-t}\cos 2t, \quad \text{for } t \geq 0 \]
Consider the following $F(s)$:

$$F(s) = \frac{s^2 + 2s + 3}{(s + 1)^3}$$

The partial-fraction expansion of this $F(s)$ involves three terms,

$$F(s) = \frac{B(s)}{A(s)} = \frac{b_1}{s + 1} + \frac{b_2}{(s + 1)^2} + \frac{b_3}{(s + 1)^3}$$

$$b_3 = \left[ (s + 1)^3 \frac{B(s)}{A(s)} \right]_{s=-1}$$

$$= (s^2 + 2s + 3)_{s=-1}$$

$$= 2$$

$$b_2 = \left\{ \frac{d}{ds} \left[ (s + 1)^3 \frac{B(s)}{A(s)} \right] \right\}_{s=-1}$$

$$= (2s + 2)_{s=-1}$$

$$= 0$$

$$b_1 = \frac{1}{2!} \left\{ \frac{d^2}{ds^2} \left[ (s + 1)^3 \frac{B(s)}{A(s)} \right] \right\}_{s=-1}$$

$$= \frac{1}{2} \left( 2 \right) = 1$$

We thus obtain

$$f(t) = \mathcal{L}^{-1}[F(s)]$$

$$= \mathcal{L}^{-1}\left[ \frac{1}{s + 1} \right] + \mathcal{L}^{-1}\left[ \frac{0}{(s + 1)^2} \right] + \mathcal{L}^{-1}\left[ \frac{2}{(s + 1)^3} \right]$$

$$= e^{-t} + 0 + t^2 e^{-t}$$

$$= (1 + t^2)e^{-t}, \quad \text{for} \quad t \geq 0$$

Solving linear D.E. using Laplace transform

Find the solution $x(t)$ of the differential equation

$$x + 3x + 2x = 0, \quad x(0) = a, \quad \dot{x} = b$$

where $a$ and $b$ are constants.
By writing the Laplace transform of \( x(t) \) as \( X(s) \) or

\[
\mathcal{L}[x(t)] = X(s)
\]

we obtain

\[
\mathcal{L}[x] = sX(s) - x(0)
\]

\[
\mathcal{L}[\dot{x}] = s^2X(s) - sx(0) - \dot{x}(0)
\]

And so the given differential equation becomes

\[
(s^2X(s) - sx(0) - \dot{x}(0)) + 3(sX(s) - x(0)) + 2X(s) = 0
\]

By substituting the given initial conditions into this last equation, we obtain

\[
(s^2 + 3s + 2)X(s) = as + b + 3a
\]

Solving for \( X(s) \), we have

\[
X(s) = \frac{as + b + 3a}{s^2 + 3s + 2} = \frac{a + b}{s + 1} + \frac{2a + b}{s + 2}
\]

The inverse Laplace transform of \( X(s) \) gives

\[
x(t) = \mathcal{L}^{-1}[X(s)] = \mathcal{L}^{-1}\left[\frac{2a + b}{s + 1}\right] - \mathcal{L}^{-1}\left[\frac{a + b}{s + 2}\right]
\]

\[
= (2a + b)e^{-t} - (a + b)e^{-2t}, \quad \text{for } t \geq 0
\]

Find the solution \( x(t) \) of the differential equation

\[
x + 2\dot{x} + 3X - 3, \quad x(0) = 0, \quad \dot{x}(0) = 0
\]

Noting that \( \mathcal{L}[3] = \frac{3}{s} \), \( x(0) = 0 \), and \( \dot{x}(0) = 0 \), the Laplace transform of the differential equation becomes

\[
s^2X(s) + 2sX(s) + 5X(s) = \frac{3}{s}
\]

Solving for \( X(s) \), we find

\[
X(s) = \frac{\frac{3}{s}}{s^2 + 2s + 5} = \frac{3}{5s} - \frac{3}{10} \frac{s + 2}{(s + 1)^2 + 2^2}
\]

\[
X(s) = \frac{3}{5} \left( \frac{1}{s} - \frac{2}{(s + 1)^2 + 2^2} \right) - \frac{3}{5} \left( \frac{s + 1}{(s + 1)^2 + 2^2} \right)
\]

Hence the inverse Laplace transform becomes

\[
x(t) = \mathcal{L}^{-1}[X(s)]
\]

\[
= \frac{3}{5} e^{-t} \sin 2t - \frac{3}{5} e^{-t} \cos 2t, \quad \text{for } t \geq 0
\]
Find the Laplace transform of \( f(t) \) defined by
\[
f(t) = 0, \quad \text{for } t < 0
\]
\[
= te^{-\alpha t}, \quad \text{for } t \geq 0
\]

**Solution.** Since
\[
\mathcal{L}[f(t)] = G(s) = \frac{1}{s^2}
\]
referring to Equation (2-6), we obtain
\[
F(s) = \mathcal{L}[te^{-\alpha t}] = G(s + \alpha) = \frac{1}{(s + \alpha)^2}
\]

What is the Laplace transform of
\[
f(t) = 0, \quad \text{for } t < 0
\]
\[
= \sin(\omega t + \theta), \quad \text{for } t \geq 0
\]

where \( \theta \) is a constant?

**Solution.** Noting that
\[
\sin(\omega t + \theta) = \sin \omega t \cos \theta + \cos \omega t \sin \theta
\]
we have
\[
\mathcal{L}[\sin(\omega t + \theta)] = \cos \theta \mathcal{L}[\sin \omega t] + \sin \theta \mathcal{L}[\cos \omega t]
\]
\[
= \cos \theta \frac{\omega}{s^2 + \omega^2} + \sin \theta \frac{s}{s^2 + \omega^2}
\]
\[
= \frac{\omega \cos \theta + s \sin \theta}{s^2 + \omega^2}
\]

Find Laplace transform \( F(S) \) of the function \( f(t) \) shown in Fig. 2.9

**Solution.** The function \( f(t) \) can be written
\[
f(t) = \frac{1}{a^2} 1(t) - \frac{2}{a^2} 1(t - a) + \frac{1}{a^2} 1(t - 2a)
\]
Then
\[
F(s) = \mathcal{L}[f(t)]
\]
\[
= \frac{1}{a^2} \mathcal{L}[1(t)] - \frac{2}{a^2} \mathcal{L}[1(t - a)] + \frac{1}{a^2} \mathcal{L}[1(t - 2a)]
\]
\[
= \frac{1}{a^2} \frac{1}{s} - \frac{2}{a^2} \frac{1}{s} e^{-as} + \frac{1}{a^2} \frac{1}{s} e^{-2as}
\]
\[
= \frac{1}{a^2} \left( 1 - 2e^{-as} + e^{-2as} \right)
\]
## Laplace Transform Table

<table>
<thead>
<tr>
<th>Laplace Transform $F(s)$</th>
<th>Time Function $f(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1$</td>
<td>Unit-impulse function $\delta(t)$</td>
</tr>
<tr>
<td>$1$</td>
<td>Unit-step function $u(t)$</td>
</tr>
<tr>
<td>$\frac{1}{s}$</td>
<td>Unit-ramp function $t$</td>
</tr>
<tr>
<td>$\frac{1}{s^2}$</td>
<td>$t^2(n = positive integer)$</td>
</tr>
<tr>
<td>$\frac{1}{s^n + 1}$</td>
<td>$t^n(n = positive integer)$</td>
</tr>
<tr>
<td>$\frac{1}{s + \alpha}$</td>
<td>$e^{-\alpha t}$</td>
</tr>
<tr>
<td>$\frac{1}{(s + \alpha)^2}$</td>
<td>$te^{-\alpha t}$</td>
</tr>
<tr>
<td>$\frac{n!}{(s + \alpha)^n - 1}$</td>
<td>$t^n e^{-\alpha t}(n = positive integer)$</td>
</tr>
<tr>
<td>$\frac{1}{(s + \alpha)(s + \beta)}$</td>
<td>$\frac{1}{\beta - \alpha}(e^{-\alpha t} - e^{-\beta t})(\alpha \neq \beta)$</td>
</tr>
<tr>
<td>$\frac{s}{(s + \alpha)(s + \beta)}$</td>
<td>$\frac{1}{\beta - \alpha}(\beta e^{-\beta t} - \alpha e^{-\alpha t})(\alpha \neq \beta)$</td>
</tr>
<tr>
<td>$\frac{1}{s(s + \alpha)}$</td>
<td>$\frac{1}{\alpha}(1 - e^{-\alpha t})$</td>
</tr>
<tr>
<td>$\frac{1}{s(s + \alpha)^2}$</td>
<td>$\frac{1}{\alpha^2}(1 - e^{-\alpha t} - \alpha e^{-\alpha t})$</td>
</tr>
<tr>
<td>$\frac{1}{s^2(s + \alpha)}$</td>
<td>$\frac{1}{\alpha^2}(\alpha t - 1 + e^{-\alpha t})$</td>
</tr>
<tr>
<td>$\frac{1}{s^2(s + \alpha)^2}$</td>
<td>$\frac{1}{\alpha^2}\left[t - \frac{2}{\alpha} + \left(t + \frac{2}{\alpha}\right)e^{-\alpha t}\right]$</td>
</tr>
<tr>
<td>$\frac{s}{(s + \alpha)^2}$</td>
<td>$(1 - \alpha t)e^{-\alpha t}$</td>
</tr>
<tr>
<td>Laplace Transform $F(s)$</td>
<td>Time Function $f(t)$</td>
</tr>
<tr>
<td>--------------------------</td>
<td>----------------------</td>
</tr>
<tr>
<td>$\omega_n$</td>
<td>$\sin \omega_n t$</td>
</tr>
<tr>
<td>$s$</td>
<td>$\cos \omega_n t$</td>
</tr>
<tr>
<td>$\frac{\omega_n^2}{s(\frac{1}{s} + \omega_n^2)}$</td>
<td>$1 - \cos \omega_n t$</td>
</tr>
<tr>
<td>$\frac{\omega_n^2(s + \alpha)}{s^2 + \omega_n^2}$</td>
<td>$\omega_n \sqrt{\alpha^2 + \omega_n^2} \sin(\omega_n t + \theta)$ where $\theta = \tan^{-1}(\omega_n/\alpha)$</td>
</tr>
<tr>
<td>$\frac{\omega_n}{(s + \alpha)(s^2 + \omega_n^2)}$</td>
<td>$\frac{\omega_n e^{-\alpha t}}{\alpha^2 + \omega_n^2} e^{\alpha t} + \frac{1}{\sqrt{\alpha^2 + \omega_n^2}} \sin(\omega_n t - \theta)$ where $\theta = \tan^{-1}(\omega_n/\alpha)$</td>
</tr>
<tr>
<td>$\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$</td>
<td>$\omega_n \sqrt{1 - \zeta^2} \sin(\omega_n \sqrt{1 - \zeta^2} t + \theta)$ where $\theta = \cos^{-1}\zeta$ ($\zeta &lt; 1$)</td>
</tr>
<tr>
<td>$\frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$</td>
<td>$\frac{1}{\sqrt{1 - \zeta^2}} e^{-\omega_n t} \sin(\omega_n \sqrt{1 - \zeta^2} t + \theta)$ where $\theta = \cos^{-1}\zeta$ ($\zeta &lt; 1$)</td>
</tr>
<tr>
<td>$\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$</td>
<td>$\frac{-\omega_n}{\sqrt{1 - \zeta^2}} e^{-\omega_n t} \sin(\omega_n \sqrt{1 - \zeta^2} t - \theta)$ where $\theta = \cos^{-1}\zeta$ ($\zeta &lt; 1$)</td>
</tr>
<tr>
<td>$\frac{\omega_n^2(s + \alpha)}{s^2 + 2\zeta\omega_n s + \omega_n^2}$</td>
<td>$\omega_n \sqrt{\alpha^2 - 2\alpha \zeta \omega_n + \omega_n^2} e^{-\omega_n t} \sin(\omega_n \sqrt{1 - \zeta^2} t + \theta)$ where $\theta = \tan^{-1}(\omega_n/\alpha - \zeta \omega_n)$ ($\zeta &lt; 1$)</td>
</tr>
<tr>
<td>$\omega_n^2$</td>
<td>$t - \frac{2\zeta}{\omega_n} \frac{1}{\omega_n \sqrt{1 - \zeta^2}} e^{-\omega_n t} \sin(\omega_n \sqrt{1 - \zeta^2} t + \theta)$ where $\theta = \cos^{-1}(2\zeta^2 - 1)$ ($\zeta &lt; 1$)</td>
</tr>
</tbody>
</table>
3. Second order systems

Any 2\textsuperscript{nd} order control system can be represented in general as shown in Fig. 2.9.

![Second-order system diagram](image)

The closed-loop transfer function \( \frac{C(s)}{R(s)} \) can be given by:

\[
\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}
\]

This form is called the standard form of the second-order system.

The dynamic behavior of the second-order system can then be described in terms of two parameters, which are the damping ratio (\( \zeta \)) and the undamped natural frequency (\( \omega_n \)).

If \( 0 < \zeta < 1 \), the closed-loop poles are complex conjugates and lie in the left-half side of the \( S \) plane. The system is then called underdamped, and the transient response is oscillatory. If \( \zeta = 0 \), the oscillation does not die out, and the system is called undamped system or continuous oscillatory system. The closed-loop poles lie on the imaginary axis. If \( \zeta = 1 \), the system is called critically damped and the system has no oscillation and the closed-loop poles are equal and real.

The dynamic response at different values of \( \zeta \) is shown in Fig. 2.10. From that figure, we see that an underdamped system with \( \zeta \) between 0.5 and 0.8 gets close to the final value more rapidly than a critically damped system. Also we can see the continuous oscillation of the system when \( \zeta \) equal zero.

![Dynamic behavior of the 2\textsuperscript{nd} order system at different damping ratios](image)
1- For the following waveforms, find the function \( f(t) \), then calculate \( F(S) \).

\begin{align*}
\text{Waveform 1} & : f(t) = 3e^{-t} - e^{-2t} \\
\text{Waveform 2} & : f(t) = 2e^{-t} \cos(10t) - t^4 + 6e^{-t-10} \\
\text{Waveform 3} & : f(t) = \cos[2(t-1)] + \sin[2(t-1)] \\
\text{Waveform 4} & : f(t) = e^{-4t} + \sin(t-2) + t^2 e^{-2t}
\end{align*}

2- Find the function \( F(S) \) of the following systems:

a) \( f(t) = 3e^{-t} - e^{-2t} \)

b) \( f(t) = 2e^{-t} \cos(10t) - t^4 + 6e^{-t-10} \)

c) \( f(t) = \cos[2(t-1)] + \sin[2(t-1)] \)

d) \( f(t) = e^{-4t} + \sin(t-2) + t^2 e^{-2t} \)

3- Find the function \( f(t) \) using Laplace transform tables of the following systems:

a) \( F(S) = \frac{1}{S(S+1)} \)

b) \( F(S) = \frac{2(S+1)}{S(S+3)(S+5)^2} \)

c) \( F(S) = \frac{S}{(S+2)(S+1)^2} \)

d) \( F(S) = \frac{(S+3)(S+4)(S+5)}{(S+2)(S+1)} \)

e) \( F(S) = \frac{10}{(S+4)(S+1)^3} \)

4- a) Find the solution of the control system described by the following differential equation (D.E.):

\[
\frac{d^3 y(t)}{dt^3} + 3 \frac{d^2 y(t)}{dt^2} - \frac{dy(t)}{dt} + 6y(t) = \frac{d^2 x(t)}{dt^2} - x(t)
\]
Where $y(t)$ and $x(t)$ are the system output and input, respectively. Also, $y(0) = 0$ 
& $\frac{dy(t)}{dt} = 0$ and $\frac{d^2 y(t)}{dt^2} = 1$

b) Find the free response and the forced response of the system.

5- For the second-order system whose output $y(t)$ is and input $x(t)$ is and described by the D.E. :

\[\ddot{y}(t) + 5\dot{y}(t) + 9y(t) = 9x(t)\]

Find

- Undamped Natural Frequency ($\omega_n$)
- Damping Ratio ($\zeta$)
- Damped Natural Frequency ($\omega_d$)
- Time Constant ($\tau$)

6- Find the poles and zeros of the system given in problem (3), Mark the poles with "X" and the zeros with "O" in the s-plane.

7- A closed-loop control system whose Transfer Function (T.F.) is:

\[\frac{C(S)}{R(S)} = \frac{5A}{S^2 + 34.5S + 5A}\]

Assuming $A = 200$, find the time response $C(t)$ when applying a unit step function as an input. Calculate the output value at $t = 3$ sec.

8- A closed-loop control system whose Transfer Function (T.F.) is:

\[\frac{C(S)}{R(S)} = \frac{2S + 1}{S^2 + 2S + 1}\]

Find the time response $C(t)$ when applying a unit step function as an input. Calculate the output value at $t = 6$ sec.